Instructions to candidates

Answer all of section A and THREE questions from section B. The total of marks available on Section A is 55.

In this paper $\mathbf{i}, \mathbf{j}$ and $\mathbf{k}$ represent unit vectors parallel to the x, y and z axes respectively. The magnitude of a vector $\mathbf{A}$ is denoted A.

SECTION A

1. Find all the eigenvalues of the matrix

$$\mathbf{A} = \begin{pmatrix} 9 & 8 \\ 2 & 3 \end{pmatrix}$$

and, for each eigenvalue, find an eigenvector of $\mathbf{A}$.

Write down an invertible 2×2 matrix $\mathbf{T}$ such that $\mathbf{T}^{-1}\mathbf{A}\mathbf{T}$ is a diagonal matrix. [8 marks]

2. Calculate the determinants of the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{A}^{-1}$ and $\mathbf{AB}$, where

$$\mathbf{A} = \begin{pmatrix} 2 & 4 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 3 \end{pmatrix} \qquad \text{and} \qquad \mathbf{B} = \begin{pmatrix} 1 & 0 & 2 \\ 2 & -1 & 2 \\ 1 & 1 & 0 \end{pmatrix}.$$

[8 marks]

3. Show that the matrix

$$\begin{pmatrix} \frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{3}} \\ -\frac{2}{\sqrt{6}} & 0 & \frac{1}{\sqrt{3}} \end{pmatrix}$$

is an orthogonal matrix. [4 marks]

4. A particle moves in a helical trajectory given by the equation:

$$\mathbf{r} = b \sin \omega t \mathbf{i} + b \cos \omega t \mathbf{j} + ct^2 \mathbf{k},$$

where b, c and ω are constants. Calculate $\dot{\mathbf{r}}$ and $\ddot{\mathbf{r}}$, and show that the magnitude of the acceleration of the particle is constant. [5 marks]

5. A uniform thin rod has mass M and length $2a$. Calculate the moment of inertia of the rod I about an axis perpendicular to the rod through its mid point (centre of mass).

If the rod rotates with constant angular velocity ω about this axis, give expressions in terms of M, a and ω for the kinetic energy and angular momentum associated with the motion. [10 marks]

6. Calculate

(i) the gradient of the scalar field

$$\phi = x^2 y^2,$$

(ii) the curl of the vector field

$$\mathbf{F} = axy^2\mathbf{i} + 2x^2y\mathbf{j} + z^3\mathbf{k},$$

where a is a constant, and

(iii) the value of a for which $\mathbf{F}$ is conservative. [6 marks]

7. Evaluate the double integral

$$\int\int_S x\,dx\,dy$$

where S is the triangular region bounded by the lines $y = 0$, $x = 1$ and $y = x$. [4 marks]

8. Evaluate the line integral $\int_C \mathbf{F}.d\mathbf{r}$ where $\mathbf{F} = 3xy\mathbf{i} - y^2\mathbf{j}$ and C is the curve in the xy plane, $y = 2x^2$, from $(0, 0)$ to $(1, 2)$. [6 marks]

9. State Stokes's theorem for vector integrals. Starting from the Maxwell equation

$$\nabla \wedge \mathbf{B} = \mu_0 \mathbf{J},$$

(valid for steady current density $\mathbf{J}$), use Stokes's theorem to establish Ampère's law:

$$\int_C \mathbf{B}.\mathbf{dr} = \mu_0 I$$

where C is a closed loop, and I is the total current through a surface bounded by C. [4 marks]

SECTION B

10. Let $\mathbf{A}$ be the matrix

$$\mathbf{A} = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 2 & -2 \\ 0 & -2 & 5 \end{pmatrix}.$$

Find a 3×3 orthogonal matrix $\mathbf{P}$ such that

$$\mathbf{P}^T\mathbf{A}\mathbf{P} = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix},$$

where $\lambda_1, \lambda_2, \lambda_3$ are the eigenvalues of $\mathbf{A}$.

(i) Show that the surface with equation

$$3x^2 + 2y^2 + 5z^2 - 4yz = 18$$

is an ellipsoid, and give the lengths of its semi-major axis and semi-minor axes and the directions of its principal axes.

(ii) The rotational kinetic energy of a certain rigid body of mass M about its centre of mass, G, is given with respect to Cartesian coordinates $Gxyz$ at G by the formula

$$T_{rot} = \frac{1}{2}Ma^2\boldsymbol{\omega}^T\mathbf{A}\boldsymbol{\omega},$$

where $\boldsymbol{\omega}$ is the angular velocity vector relative to these axes and a is a constant. Give the moments of inertia about each of the principal axes of the body.
[15 marks]

11. The motion of a projectile in the earth's gravitational field is described to leading order in the earth's angular velocity $\boldsymbol{\omega}$ by the equation

$$\ddot{\mathbf{r}} = \mathbf{g} - 2\boldsymbol{\omega} \wedge \dot{\mathbf{r}} \qquad (1)$$

where $\mathbf{g}$ is the acceleration due to gravity, (which may be assumed to be constant) and $\mathbf{r}$ is the position vector of the particle relative to axes fixed in the earth. Choosing axes so that $\mathbf{i}$ is east, and $\mathbf{j}$ is north, show with the aid of a diagram that:

$$\boldsymbol{\omega} = \omega \sin\theta \mathbf{j} + \omega \cos\theta \mathbf{k} \quad \text{and}$$

$$\mathbf{g} = -g\mathbf{k},$$

where $\theta = \pi/2 - \lambda$, λ being the latitude of the projectile.

Hence show that in components Equation (1) becomes

$$\ddot{x} = -2\omega(\dot{z}\sin\theta - \dot{y}\cos\theta) \qquad (2a)$$

$$\ddot{y} = -2\omega\cos\theta\dot{x} \qquad (2b)$$

$$\ddot{z} = -g + 2\omega\sin\theta\dot{x}. \qquad (2c)$$

An object is dropped from rest at time $t = 0$ and a height h above the ground. Show that if ω is neglected, it falls vertically so that $\dot{z} = -gt$ and $z = h - gt^2/2$.

Then show that in fact the object hits the ground a distance east of its starting point given to leading order in ω by

$$\delta x = \frac{1}{3}\omega\sin\theta\sqrt{\frac{8h^3}{g}}.$$

[15 marks]

12. The motion of a particle of mass m in a central potential $V(r)$ satisfies the following equations:

$$\begin{aligned} \tfrac{1}{2}m\dot{r}^2 + \frac{J^2}{2mr^2} + V(r) &= E \\ J &= mr^2\dot{\theta} \end{aligned} \qquad (1)$$

where J and E are constants; comment briefly on their physical significance.

Show that

$$\frac{dr}{d\theta} = \frac{mr^2}{J}\dot{r}$$

and hence that:

$$\left(\frac{dr}{d\theta}\right)^2 + r^2 = \frac{2mr^4}{J^2}\left[E - V\right].$$

If $V(r) = -\alpha/r^4$, where $\alpha > 0$, show that for $E = 0$ the trajectory of the particle is given by

$$r = r_0 \sin(\theta - \theta_0)$$

where r_0 is a function of J, m and α which you should determine, and θ_0 is an arbitrary constant.

Sketch the trajectory for $\theta_0 = 0$, and discuss what happens in that case as $\theta \to 0$. [15 marks]

13. A particle of mass m subject to a force $\mathbf{F}$ moves in a plane. Starting from the equations

$$\begin{aligned} \hat{\mathbf{r}} &= \cos\theta\mathbf{i} + \sin\theta\mathbf{j} \\ \hat{\boldsymbol{\theta}} &= -\sin\theta\mathbf{i} + \cos\theta\mathbf{j}, \end{aligned}$$

where $\hat{\mathbf{r}}$ and $\hat{\boldsymbol{\theta}}$ are unit radial and transverse vectors respectively, show that in polar coordinates (r, θ) the equation of motion $\mathbf{F} = m\ddot{\mathbf{r}}$ becomes

$$\begin{aligned} m(\ddot{r} - r\dot{\theta}^2) &= F_r, \\ m(r\ddot{\theta} + 2\dot{r}\dot{\theta}) &= F_\theta \end{aligned}$$

where $\mathbf{F} = F_r\hat{\mathbf{r}} + F_\theta\hat{\boldsymbol{\theta}}$.

A small uniform cylinder of radius R rolls without slipping along the inside of a fixed cylinder of radius $r >> R$. The axes of symmetry of the two cylinders are parallel. Assuming only small excursions from the equilibrium position, show that the motion of the small cylinder is simple harmonic, and that the frequency of oscillation is that of a simple pendulum of length

$$L = \frac{3}{2}(r - R).$$

[15 marks]

14. The divergence theorem for vector integrals states that for any vector field $\mathbf{A}$,

$$\int_S \mathbf{A}.\mathbf{dS} = \int_V \mathrm{div}\mathbf{A}\, dV,$$

where the volume V is enclosed by the surface S.

Given $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, calculate $\mathrm{div}\,\mathbf{r}$ and $\mathrm{div}\, r\mathbf{r}$. Verify the divergence theorem for the following cases:

(i) $\mathbf{A} = \mathbf{r}$, and S a sphere of radius a, centre the origin;

(ii) $\mathbf{A} = \mathbf{r}$, and S a cube with sides of length a, choosing the origin at one corner of the cube and axes along the three edges meeting at the origin;

(iii) $\mathbf{A} = r\mathbf{r}$, and S a sphere of radius a, centre the origin.

(For a vector field of the form $\mathbf{A} = f(r)\mathbf{r}$,

$$\mathrm{div}\,\mathbf{A} = \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^3 f(r)\right] .)$$

[15 marks]