SECTION A

1. Find all solutions (if there are any) of the system of equations

$$\begin{aligned} x + 3y + z &= 2 \\ y + 3z &= 3 \\ -x - 2y + 2z &= 1 \\ 2x + 5y - z &= 1. \end{aligned}$$

[5 marks]

2. Find all solutions (if there are any) of the system of equations

$$\begin{aligned} x - z + 2t &= 0 \\ 3x + y - 2z + 5t &= 1 \\ x + 3y + 2z - t &= 3. \end{aligned}$$

[5 marks]

3. Find the inverse of the matrix

$$\begin{pmatrix} 1 & 2 & 1 \\ 3 & 5 & -1 \\ -1 & 0 & 8 \end{pmatrix},$$

and check your answer. [6 marks]

4. Find the determinant of the matrix

$$A = \begin{pmatrix} 1 & 3 & 2 & 1 \\ 3 & 0 & 5 & 8 \\ 4 & 0 & 6 & 7 \\ 2 & 3 & 4 & 2 \end{pmatrix}.$$

[5 marks]

5. Let

$$\mathbf{u} = 2\mathbf{i} - 3\mathbf{j} - 4\mathbf{k}, \quad \mathbf{a} = \mathbf{i} - \mathbf{j} - \mathbf{k}.$$

Find $\mathbf{u} \cdot \mathbf{a}$, and write $\mathbf{u}$ as the sum of a vector parallel to $\mathbf{a}$ and a vector orthogonal to $\mathbf{a}$. [4 marks]

6. Let

$$\mathbf{u} = 2\mathbf{i} + \mathbf{j} + \mathbf{k}, \quad \mathbf{v} = \mathbf{i} + 3\mathbf{j} - \mathbf{k}, \quad \mathbf{w} = -\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}.$$

Find $(\mathbf{u} \times \mathbf{v}) \times \mathbf{w}$ and verify by direct calculation that

$$(\mathbf{u} \times \mathbf{v}) \times \mathbf{w} = (\mathbf{u} \cdot \mathbf{w})\mathbf{v} - (\mathbf{v} \cdot \mathbf{w})\mathbf{u}.$$

[6 marks]

7. Find an orthonormal basis for $\mathbf{R}^3$ containing two vectors which lie in the plane

$$x + y - 2z = 0.$$

[6 marks]

8. Show that the vectors

$$(1, -3, 0, 1), \quad (3, -5, 3, 5) \quad \text{and} \quad (1, 1, 3, 3)$$

in $\mathbf{R}^4$ are linearly dependent.

Find a basis for the subspace of $\mathbf{R}^4$ spanned by these three vectors.

[6 marks]

9. Find the eigenvalues and eigenvectors of the matrix

$$\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}.$$

[6 marks]

10. Find the eigenvalues of the matrix

$$A = \begin{pmatrix} -1 & 5 & -3 \\ -1 & 6 & -4 \\ -1 & 7 & -5 \end{pmatrix}.$$

Hence write down a diagonal matrix which is similar to A. [6 marks]

SECTION B

11. Let

$$A = \begin{pmatrix} 1 & -3 & 1 \\ 2 & -5 & \alpha+2 \\ 2 & \alpha-6 & 3 \end{pmatrix}.$$

Show that $\det A = 0$ if and only if $\alpha = \pm 1$. [4 marks]

Solve the equations

$$\begin{aligned} x \; - \quad\;\; 3y \; + \qquad\quad z &= 1 \\ 2x \; - \quad\;\; 5y \; + (\alpha+2)z &= 4 \\ 2x + (\alpha-6)y \; + \qquad 3z &= 4 \end{aligned}$$

(i) when $\alpha = 1$, [3 marks]

(ii) when $\alpha = -1$, [3 marks]

(iii) when $\alpha \neq \pm 1$. [5 marks]

12. Let

$$A = (1,0,1), \quad B = (0,1,1), \quad C = (2,-1,3) \quad \text{and} \quad D = (3,4,1).$$

(i) Find the volume of the parallelepiped which has one vertex at A and the three vertices adjacent to this at B, C and D. [6 marks]

(ii) Find a vector $\mathbf{w}$ which is orthogonal to the plane BCD. [4 marks]

(iii) Find $\overrightarrow{AB} \cdot \mathbf{w}$. [2 marks]

(iv) Using (iii), or otherwise, find the distance of the point A from the plane BCD. [3 marks]

13. Let

$$A = \begin{pmatrix} 1 & 0 & 2 & 2 \\ -3 & 2 & 4 & -4 \\ 3 & -1 & 1 & 5 \end{pmatrix}.$$

(i) Find a basis for the row space of A. [4 marks]

(ii) Find a basis for the nullspace of A. [5 marks]

(iii) Find a basis for the column space of A. Extend your basis to a basis of $\mathbf{R}^3$. [6 marks]

14. The eigenvalues of the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ -1 & -1 & 0 \end{pmatrix}$$

are 2, 1 and -1.

(i) For each eigenvalue, find an eigenvector of length 1. [7 marks]

(ii) Write down an orthogonal matrix P and a diagonal matrix D such that $P^{-1}AP = D$. [3 marks]

(iii) Use (ii) to express the quadric surface

$$x^2 + y^2 - 2xz - 2yz - 2 = 0$$

in standard form, and say what type of quadric surface it is. [5 marks]