2MM23 Mathematics Foundation Unit II
January 1997

You may attempt all questions.

All answers to section A and the best THREE answers to section B will be taken into account.

The marks noted indicate the relative weight of the questions.

SECTION A

1. Find all solutions (if there are any) of the system of equations

$$\begin{aligned} x + y - z &= 1 \\ 2x + 3y + z &= 2 \\ -2x \quad\quad + z &= 1. \end{aligned}$$

[5 marks]

2. Find all solutions (if there are any) of the system of equations

$$\begin{aligned} x + y - z + t &= 1 \\ 2x + 2y + z - 2t &= 0 \\ x + y + 5z - 7t &= -3. \end{aligned}$$

[5 marks]

3. Let

$$A = \begin{pmatrix} 4 & 5 & 2 & 1 \\ 3 & 4 & 1 & 1 \\ 5 & 6 & 2 & 1 \\ 4 & 4 & 3 & 1 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} -1 & 2 & 1 & 5 \\ 0 & 3 & 4 & 0 \\ 0 & 0 & -2 & 1 \\ 0 & 0 & 0 & 2 \end{pmatrix}.$$

Find

(i) $\det A$ [4 marks]

(ii) $\det B$ [2 marks]

(iii) $\det A^{-1}B$. [2 marks]

4. Write down an equation for the plane which is normal to the vector $\mathbf{n} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and contains the point $P = (1, 1, 0)$. Determine the distance of this plane from the point $(1, 3, -1)$. [5 marks]

5. Let

$$\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}, \quad \mathbf{v} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}, \quad \mathbf{w} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}.$$

Find $\mathbf{u} \cdot (\mathbf{v} \times \mathbf{w})$. [3 marks]

Hence, or otherwise, find the volume of the parallelepiped which has one vertex at the point $A = (1, 0, 1)$ and the three vertices adjacent to this at

$$B = (2, -1, 3), \quad C = (3, 1, 4), \quad D = (4, 1, 0).$$

[3 marks]

6. Let S be the subspace of $\mathbf{R}^3$ spanned by $(2,1,2)$ and $(1,-1,4)$. Find an orthonormal basis for S, and extend this to an orthonormal basis for $\mathbf{R}^3$.
[6 marks]

7. Find the rank and nullity of the matrix

$$\begin{pmatrix} 1 & 1 & 4 & 6 & 2 & 4 & 0 \\ 0 & -1 & 3 & 2 & 0 & 1 & 1 \\ 2 & 3 & 4 & 9 & 7 & 7 & -1 \\ 1 & 2 & 1 & 4 & 2 & 3 & -1 \end{pmatrix}.$$

[6 marks]

8. Find the eigenvalues of the matrix

$$A = \begin{pmatrix} 7 & -4 \\ 12 & -7 \end{pmatrix}.$$

Hence write down a diagonal matrix which is similar to A. [5 marks]

9. Show that the matrix

$$A = \begin{pmatrix} 0 & -3 & -1 \\ 1 & 1 & 1 \\ -3 & 0 & -2 \end{pmatrix}$$

has eigenvalues 0, 1 and -2. [3 marks]

Find the eigenvectors of A corresponding to each of these eigenvalues.
[6 marks]

SECTION B

10. Let

$$A = \begin{pmatrix} 2 & \alpha - 2 & 7 \\ 1 & -1 & 3 \\ -1 & 0 & \alpha - 1 \end{pmatrix}.$$

(i) Show that A is invertible if and only if $\alpha \neq -1$.

[5 marks]

(ii) Find the inverse of A when $\alpha = 0$. [5 marks]

(iii) Find a condition which a, b and c must satisfy for the system of equations

$$\begin{aligned} 2x - 3y + 7z &= a \\ x - y + 3z &= b \\ -x - 2z &= c \end{aligned}$$

to be consistent. [5 marks]

11. Let L denote the line of intersection of the planes

$$x - y + 2z - 4 = 0 \text{ and } 2x + y + z - 2 = 0,$$

and let L' denote the line joining the points

$$A = (0, 1, 1) \quad \text{and} \quad B = (-1, 3, 1).$$

(i) Find a parametric equation for the line L. [4 marks]

(ii) Write down a vector parallel to L'. Hence, or otherwise, find parametric equations for L'. [4 marks]

(iii) Determine the point in which L' meets the plane

$$x - y + 2z - 4 = 0.$$

[4 marks]

(iv) Decide whether or not L' meets L. [3 marks]

12. Vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ in $\mathbf{R}^4$ are defined by

$$\mathbf{v}_1 = (1,2,4,2),\ \mathbf{v}_2 = (2,1,1,1),\ \mathbf{v}_3 = (-1,1,3,0),\ \mathbf{v}_4 = (3,0,-2,3).$$

(i) Show that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$ are linearly dependent. [5 marks]

(ii) Find a basis for the subspace S of $\mathbf{R}^4$ spanned by $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$. Extend your basis to a basis of $\mathbf{R}^4$. [5 marks]

(iii) Show that the vector $(2,-2,-6,1)$ lies in S. [5 marks]

13. The eigenvalues of the matrix

$$A = \begin{pmatrix} 2 & 0 & -2 \\ 0 & 0 & 2 \\ -2 & 2 & 1 \end{pmatrix}$$

are 4, 1 and -2.

(i) For each eigenvalue, find an eigenvector of length 1. [7 marks]

(ii) Write down an orthogonal matrix P with $\det P = 1$ and a diagonal matrix D such that $P^{-1}AP = D$. [4 marks]

(iii) Use (ii) to express the quadric surface

$$2x^2 + z^2 - 4xz + 4yz + 3 = 0$$

in standard form, and say what type of quadric surface it is. [4 marks]