2MA76

JUNE 1998

Full marks can be obtained for complete answers to FIVE questions.

Only the best FIVE answers will be counted.

Throughout this paper standard notation is used. Thus X, Y and Z denote population densities of susceptible, infected and immune individuals, respectively. Additionally, N or H is the total density of host individuals. Furthermore, β is the transmission parameter, γ is the rate of recovery, ν is the rate of loss of immunity, μ or b is the death rate, p and j are vaccination parameters, r is the intrinsic growth rate, α is the pathogenicity, Γ is the net rate of loss of infecteds, K is the carrying capacity and D is the diffusion constant.

1. The dynamics of an epidemic without removal are given by the equations

$$\frac{dX}{dt} = -\beta XY,$$
$$\frac{dY}{dt} = \beta XY,$$

where, initially, there are ηN infectives and $(1-\eta)N$ susceptibles with η between 0 and 1.

Find the number of susceptibles as a function of time. Find also the equation of the epidemic curve and locate and evaluate its maximum.

2. The dynamics of an epidemic with removal are given by the equations

$$\frac{dX}{dt} = -\beta XY + \nu Z,$$
$$\frac{dY}{dt} = \beta XY - \gamma Y,$$
$$\frac{dZ}{dt} = \gamma Y - \nu Z.$$

These equations also describe endemic behaviour. Find the equilibrium states and analyse these for feasibility and stability. State how you would expect the long-term outcomes to depend on parameter values.

3. With births, deaths and two vaccination protocols included, the dynamics of an epidemic are given by the equations

$$\frac{dX}{dt} = \mu N(1-p) - (\mu + j)X - \beta XY,$$
$$\frac{dY}{dt} = \beta XY - (\gamma + \mu)Y,$$
$$\frac{dZ}{dt} = \mu Np + \gamma Y + jX - \mu Z.$$

These equations also describe endemic behaviour. Find the equilibrium states and analyse these for feasibility and stability. Determine a threshold condition, involving both p and j, for the eradication of the infection in the long-term.

4. In a criss-cross venereal infection model with the removed class permanently immune and no immune individuals initially, the dynamics are given, in standard notation, by

$$\frac{dX_1}{dt} = -\beta_1 X_1 Y_2, \qquad \frac{dX_2}{dt} = -\beta_2 X_2 Y_1,$$
$$\frac{dY_1}{dt} = \beta_1 X_1 Y_2 - \gamma_1 Y_1, \qquad \frac{dY_2}{dt} = \beta_2 X_2 Y_1 - \gamma_2 Y_2,$$
$$\frac{dZ_1}{dt} = \gamma_1 Y_1, \qquad \frac{dZ_2}{dt} = \gamma_2 Y_2.$$

Describe, very briefly, the assumptions made when such a model is used.

Show that the female and male populations are both constant. Show also that

$$X_1(t) = X_1(0)\exp(-\beta_1 Z_2/\gamma_2).$$

Deduce that as, t tends to infinity, X_1 tends to a positive limit and Y_1 tends to zero. Obtain transcendental equations which determine the long-term limiting values of X_1 and X_2.

Show that the threshold condition for an epidemic to occur is that at least one of

$$X_1(0)Y_2(0)\big/Y_1(0) > \gamma_1\big/\beta_1, \qquad X_2(0)Y_1(0)\big/Y_2(0) > \gamma_2\big/\beta_2,$$

is true. What single condition would ensure an epidemic?

5. The basic dynamics of host-parasite associations in which the parasite affects host numbers are given by

$$\frac{dH}{dt} = rH - \alpha Y,$$
$$\frac{dY}{dt} = \beta XY - \Gamma Y.$$

Describe, very briefly, the assumptions made when such a model is used. Explain how the equations would need to be modified to include (separately) the following effects: (i) parasite-induced reduction of host reproduction, (ii) vertical transmission, (iii) latent periods of infections, (iv) density-dependent pathogenicity, (v) density-dependent host reproduction and (vi) transmission by free-living infective stages.

In the basic model, the pathogen is unable to regulate the host if $\alpha < r$. Show that, when this inequality holds, the host grows at a reduced exponential rate. Determine this rate ρ and the corresponding behaviour of X. Determine how these results change (if they do) when (ii) is included.

6. A spatio-temporal epidemic model is given, in dimensionless form, by the equations

$$X_t = -XY,$$
$$Y_t = XY - \lambda Y + Y_{xx},$$

where λ is positive. Explain the significance of each of the terms in these equations.

Travelling wave solutions $X(z),\ Y(z),\ z = x - ct$ are sought with

$$X(\infty) = 1,\ X'(-\infty) = Y(\infty) = Y(-\infty) = 0.$$

Explain the reason for each of these conditions.

Show that the travelling wave solution conserves the quantity

$$Y' + cY + cX - c\lambda \ln X.$$

Obtain a transcendental equation for the surviving susceptible population σ after the passage of the wavefront. Sketch σ as a function of λ and comment on your result.

7. A spatio-temporal model for the spread of rabies which includes births and deaths of the host is of the form

$$\frac{\partial X}{\partial t} = -\beta XY + rX(1 - \frac{X}{K}),$$
$$\frac{\partial Y}{\partial t} = \beta XY - bY + D\frac{\partial^2 Y}{\partial x^2}.$$

Non-dimensionalize this system to give

$$u_t = u_{xx} + uv - \lambda u,$$
$$v_t = -uv + sv(1 - v),$$

where u relates to Y and v to X. Identify λ and s.

Look for travelling wave solutions with $u > 0$ and $v > 0$ and hence show, by linearizing in the region where $v \to 1$ and $u \to 0$, that a wave may exist if $\lambda < 1$ and if so the minimum wave speed is $2(1-\lambda)^{1/2}$. What is the steady state far behind the wave?