2MA67

Instructions to candidates

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **FIVE** answers will be taken into account.

1. Define the term Markovian, with respect to a discrete time stochastic process.

A discrete time Markov process has three states, denoted 1, 2 and 3, and the transitions between them have probabilities

$$\begin{aligned} P(1 \to 2) &= \alpha, \quad P(2 \to 1) = 0 \\ P(2 \to 3) &= \beta, \quad P(3 \to 2) = 0 \\ P(3 \to 1) &= 0, \quad P(1 \to 3) = 0 \end{aligned}$$

where $\alpha \neq \beta$, $0 < \alpha < 1$ and $0\beta < 1$. Consider the resulting Markov chain and obtain the stochastic matrix Q.

Find the *right* eigenvectors and eigenvalues of Q and obtain the probability distribution as the number of steps $n \to \infty$.

The system is initially in state 1. Write down a vector describing the initial probability of the three state system. Expand this vector with respect to the three right eigenvectors as basis.

Show that the probability to be in state 3 after n steps (given that $\alpha = 1/4$ and $\beta = 3/4$) is

$$P(3, n) = 1 + \frac{1}{2}(\frac{1}{4})^n - \frac{3}{2}(\frac{3}{4})^n .$$

2. (i) Define the probability transition rate $W(x \to x')$ where x and x' are possible values of a continuous time Markov process.

Consider a stochastic process with two states 1 and 2. The transition rates are

$$W(1 \to 2) = a \quad \text{and} \quad W(2 \to 1) = b$$

where a and b are positive constants ($a \neq b$). Write down the master equations for this process.

Hence show that (with initial condition that the system is in state 1 at $t = 0$)

$$P_1(t) = \left(b + ae^{-(a+b)t}\right)/(a+b)\,.$$

(ii) Consider instead a discrete time Markov process with two states 1 and 2, and with transition probabilities

$$P(1 \to 2) = A \quad \text{and} \quad P(2 \to 1) = B$$

where A and B are positive constants ($A \neq B$). With initial condition that the system is in state 1 at $t = 0$, show that after n steps the solution is

$$P_1(n) = \left[B + A(1 - A - B)^n\right]/(A+B)\,.$$

Hence show that, by taking the limit of a large number of small steps this result is equivalent to that of part (i) of this question. [*Hint: set* $t = n\epsilon$, $A = a\epsilon$, $B = b\epsilon$ *and consider the limit* $\epsilon \to 0$.]

3. The master equation for a particular class of Markovian process can be written as

$$\frac{\partial}{\partial t}P(x,t) = \int_{-\infty}^{\infty} dr[P(x-r,t) - P(x,t)]f(r)\,,$$

where the transition rate f is given by

$$f(r) = A\exp(-r^2/b^2)$$

and A and b are positive constants. Here, the stochastic variable can take any real value x and $P(x,t)$ is the corresponding probability density function.

Show that with suitable approximations, which you should explain, the master equation reduces to the diffusion equation

$$\frac{\partial P}{\partial t} = \frac{D}{2}\frac{\partial^2 P}{\partial x^2}\,,$$

where you should find D in terms of A and b.

Verify that the probability density function

$$P(x,t) = \frac{c}{\sqrt{t}}\exp(-\lambda x^2/t)$$

satisfies this equation provided that the constants c and λ take on particular values which you should specify in terms of A and b.

[*Note: you may make use of the integral*

$$\int_{-\infty}^{\infty} dx\exp(-a^2x^2) = \frac{\sqrt{\pi}}{a}$$

and others obtained from it by differentiation with respect to a^2.]

4. Consider a sequence of n delta-function pulses at times t_i which are independent and uniformly randomly distributed in the interval 0 to T. Thus they are represented by the stochastic process

$$F(t) = \sum_{i=1}^{n} \delta(t - t_i)\,.$$

Show that

$$\overline{F(t)} = \alpha \quad \text{and}$$
$$\overline{F(t)F(t')} = \alpha^2 + \alpha\delta(t - t')\,,$$

when the limit $T \to \infty$ with $\alpha = n/T$ fixed is taken.

This stochastic force $F(t)$ acts in a Langevin equation

$$\frac{dV}{dt} = -\gamma V + F(t)\,,$$

where γ is a non-stochastic constant. Evaluate $\overline{V(t)}$ and $\overline{V^2(t)}$ given initial condition $V = 0$ at $t = 0$.

5. A system consists of a large number N of distinguishable weakly interacting particles and has total energy U. If each particle has allowed energies ϵ_j ($j = 1, 2 \ldots M$) which are non-degenerate, derive the probability that a particle is in a particular one of these energy levels.

[You may use the result that $\log N! \approx N \log N - N$ for large N.]

Write down the constraint equations for the total energy and number of particles for two such systems in thermal equilibrium. Use these constraints to derive the Boltzmann distribution for the energy of a particle in the above system of N particles when they are in thermal equilibrium at temperature T.

Show that the average energy of any particle in this system is given by

$$\overline{\epsilon} = \frac{\partial}{\partial\beta} \ln Z$$

where Z is the partition function, which you should define, and β is a function of temperature and Boltzmann's constant k.

Consider such a system with three allowed energy levels given by $\epsilon_j = 0, \pm\epsilon$. Evaluate Z, and the average energy per particle $\overline{\epsilon}$.

Give the high and low temperature limits of the average particle energy for this three level system.

6. Consider an Ising model with 4 sites and with energy

$$E(\{s\}) = -J \sum_{m=1}^{4} s_m s_{m+1}$$

where $s_m = \pm 1$ and $s_5 \equiv s_1$. Show that in thermal equilibrium at temperature T, the partition function Z is given by

$$2e^{4J/kT} + 12 + 2e^{-4J/kT} \, .$$

Find the probability that the system is in state $(+1, +1, -1, -1)$.

Consider a modified model with link 1 to 4 ignored, so with energy

$$E(\{s\}) = -J \sum_{m=1}^{3} s_m s_{m+1}$$

and find an expression for the probability that this system is in the state

$$(+1, +1, -1, -1) \, .$$

Obtain an expression for the average energy of the modified system and find the limit of this as the temperature tends to zero.

7. Specify mathematically the Hopfield model of a neural network.

Explain the relationship of this model to a statistical mechanics system at equilibrium at temperature T.

Consider a Hopfield model with 6 neurons which has been trained by the Hebb rule with pattern $(+1, +1, +1, -1, -1, -1)$. If the system is initially in state $(-1, +1, +1, -1, -1, -1)$, discuss the probability for the next state of the network.