2MA66 Relativity Summer 1997

Instructions to candidates.

Full marks can be obtained for complete answers to **FIVE** questions. Only the best **FIVE** answers will be counted.

The following results may be used freely as required

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} (g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu})$$

$$R^{\mu}{}_{\nu\sigma\rho} = \Gamma^{\mu}_{\nu\rho,\sigma} - \Gamma^{\mu}_{\nu\sigma,\rho} + \Gamma^{\mu}_{\alpha\sigma}\Gamma^{\alpha}_{\nu\rho} - \Gamma^{\mu}_{\alpha\rho}\Gamma^{\alpha}_{\nu\sigma}$$

$$R_{\mu\nu} = R^{\sigma}{}_{\mu\sigma\nu} \quad , \quad R = R^{\mu}{}_{\mu}$$

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$$c = 2.998 \times 10^{8} \text{ ms}^{-1}$$

1. Write down the Lorentz transformation relating the spacetime coordinates in a frame S and a frame S' moving at speed v along the x-axis in S. Hence deduce that moving objects undergo length contraction.

If S'' is another frame which is moving at speed w relative to S' in the positive x'-direction, find the velocity which the frame S'' moves relative to S.

Two trains of rest length L approach each other along parallel tracks. The one moving to the right travels at speed v whilst the other travels at speed αv on the other track. How long is each train in the frame of the other? What is the time between the front of one train meeting the front of the other and the rear ends passing each other as a function of L, α and v?

In the case when $\alpha = 1 + \epsilon$ and ϵ is small ($\epsilon << 1$) deduce that this time is approximately

$$\frac{L}{v}\left[1 - \frac{\epsilon}{2}\right] .$$

2. Define the momentarily comoving reference frame, (MCRF), the 4-velocity U^μ and the 4-momentum P^μ of a particle of mass m, giving the values of U^μ and P^μ in the MCRF.

The worldline of a particle in some inertial frame S is given by the parametric equations

$$(ct, x, y, z) = \left(ca \sinh\left(\frac{\lambda}{a}\right), 0, 0, ca \cosh\left(\frac{\lambda}{a}\right)\right)$$

where λ is a parameter and a is constant. Compute the 4-velocity and 4-momentum of the particle and deduce that λ is the proper time and that the acceleration is uniform. Show that its constant value in the MCRF is c/a.

If $c/a = 5\ \mathrm{ms}^{-2}$, determine the speed of the particle as a percentage of c, after 5 years have elapsed in the MCRF. How many years in S will have elapsed to reach this speed?

3. A particle of mass m_1, energy E and momentum p collides with a particle of mass m_2 which is at rest. Two particles of rest masses m_3 and m_4 emerge from the collision in directions at angles θ and ϕ respectively to the direction travelled by the first particle. Show that

$$c^2 p_3 p_4 \cos(\theta + \phi) = (E_3 - m_2 c^2)(E_4 - m_2 c^2) - \frac{1}{2} M^2 c^4$$

where the respective 4-momenta are $(E_3/c, \mathbf{p}_3)$, $(E_4/c, \mathbf{p}_4)$ and

$$M^2 = m_1^2 + m_2^2 - m_3^2 - m_4^2 .$$

In the case where the particles emerge at the same angle and are identical, determine an expression for $\cos^2\theta$ in terms of the masses and E. Deduce the scattering angle when all masses are equal and $E = 5 m_1 c^2$.

4. Consider Cartesian and plane polar coordinates (x, y) and (r, θ) respectively, where $x = r\cos\theta$ and $y = r\sin\theta$. Denoting polar coordinates with primes and Cartesian coordinates without primes, determine the transformation matrix $\Lambda^{\mu}{}_{\mu'}$ and find its inverse $\Lambda^{\mu'}{}_{\mu}$.

If e_μ and $e_{\mu'}$ are the respective bases find g_{rr}, $g_{r\theta}$ and $g_{\theta\theta}$.

The only non-zero components of the Christoffel symbols, $\Gamma^{\mu}_{\nu\sigma}$, are

$$\Gamma^{r}_{\theta\theta} = -r \quad , \quad \Gamma^{\theta}_{\theta r} = \Gamma^{\theta}_{r\theta} = \frac{1}{r} \ .$$

If V^μ is a vector with Cartesian coordinates $((x^2 - y^2)/2, xy)$, show that

$$V^{r}{}_{;r} = r\cos\theta \quad , \quad V^{r}{}_{;\theta} = -r^2\sin\theta$$

and find $V^{\theta}{}_{;r}$ and $V^{\theta}{}_{;\theta}$.

Compute $(V^{r}{}_{;r\theta} - V^{r}{}_{;\theta r})$ and hence without further calculation write down the value of

$$(V^{\theta}{}_{;r\theta} - V^{\theta}{}_{;\theta r})$$

clearly stating your reasoning.

5. The line element on a two dimensional surface is given by

$$dl^2 = y^n dx^2 + x^m dy^2$$

where m and n are non-negative integers. Write down the metric tensor, its inverse and compute all the Christoffel symbols.

Stating clearly any general symmetry properties of the Riemann tensor $R_{\mu\nu\sigma\rho}$ demonstrate that for two dimensional surfaces it possesses one independent component. Show that

$$R^{x}{}_{yxy} = -\frac{n(n-2)}{4y^2} - \frac{m(m-2)x^{m-2}}{4y^n} \ .$$

Hence determine the three independent components of the Ricci tensor and the Ricci scalar. For which values of m and n does R vanish for all x and y?

6. The metric for a weak gravitational field described by the Newtonian gravitational potential $\phi = \phi(x_i)$ is given by

$$ds^2 \;=\; (1+2\phi)dt^2 \;-\; (1-2\phi)(dx^2 \;+\; dy^2 \;+\; dz^2)$$

where $c = 1$. Write down $g_{\mu\nu}$ and deduce $g^{\mu\nu}$, ignoring terms of second order in ϕ and higher.

From the definition of the connection $\Gamma^{\mu}_{\nu\sigma}$ deduce that the only non-zero components are Γ^{i}_{jk}, Γ^{0}_{i0} and Γ^{i}_{00} and compute them to order ϕ. Hence show that

$$R^{0}{}_{i0j} \;=\; -\,\phi_{,ij} \;.$$

If the spatial components of the curvature tensor are to first order in ϕ,

$$R^{i}{}_{jkl} \;=\; -\,\delta^{i}{}_{l}\phi_{,jk} \;+\; \delta^{im}\delta_{jl}\phi_{,mk} \;+\; \delta^{i}{}_{k}\phi_{,jl} \;-\; \delta^{im}\delta_{jk}\phi_{,ml}$$

find R_{ij} and R_{00}.

If these are the only non-zero components of $R_{\mu\nu}$, determine the Einstein tensor and comment on the solution to the equation

$$G_{\mu\nu} \;=\; 0 \;.$$

7. The metric for a Schwarzschild spacetime is given by

$$ds^2 \;=\; \left(1-\frac{2\tilde{M}}{r}\right)dt^2 \;-\; \left(1-\frac{2\tilde{M}}{r}\right)^{-1} dr^2 \;-\; r^2 d\theta^2 \;-\; r^2\sin^2\theta\, d\phi^2$$

with $c = 1$. With respect to the coordinate system $x^{\mu} = (t, r, \theta, \phi)$, write down any quantities which are conserved on free particle trajectories, stating your reasoning.

A particle of non-zero mass m freely moves in this spacetime along a world line $x^{\mu}(\tau)$ in the equatorial plane, $\theta = \pi/2$. The components p_t, p_θ and p_ϕ are constant and given by

$$p_t \;=\; m\tilde{E} \quad , \quad p_\theta \;=\; 0 \quad , \quad p_\phi \;=\; -\,m\tilde{L}$$

where $p^{\mu} \;=\; m\frac{dx^{\mu}}{d\tau}$.

Determine p^t and p^{ϕ} and hence find the effective potential $\tilde{V}(r)$ of the radial motion defined by

$$\left(\frac{dr}{d\tau}\right)^2 \;=\; \tilde{E}^2 \;-\; \tilde{V}^2(r) \;.$$

If $L = 5\sqrt{3}$ and $\tilde{M} = 2$, show that for the stable circular orbit

$$\tilde{E} \;=\; \frac{13}{6\sqrt{5}} \;.$$