2MA63 JAN 1998

Instructions to candidates

Full marks can be obtained for complete answers to FIVE questions. Only the best five answers will be taken into account.

1. Write down the differential equation satisfied by the function $y(x)$ for which the functional

$$I[y] = \int_a^b F(x, y, y')\, dx \qquad y(a) = y_0, \quad y(b) = y_1$$

is stationary.

Show that if F is not an explicit function of y then the extremal curve satisfies the equation

$$\frac{\partial F}{\partial y'} = C$$

where C is a constant.

For the case

$$I[y] = \int_0^{\frac{1}{2}} (y' + (1 - x^2)y'^2)\, dx \qquad y(0) = 0, \quad y(\tfrac{1}{2}) = \ln 3$$

show that the extremal curve satisfies the equation

$$\frac{dy}{dx} = \frac{A}{1 - x^2}$$

where A is a constant. Hence find the extremal curve. Calculate $I[y]$ both for the extremal curve and for the straight line joining the two end points. What can you conclude about the nature of the extremum?

2. A dynamical system with one degree of freedom $q(t)$ is described by the Lagrangian $L(q, \dot{q}, t)$, where $\dot{q} \equiv \frac{dq}{dt}$. Write down Lagrange's equation.

The Hamiltonian H is defined as follows:

$$H(q, p, t) = p\dot{q} - L$$

where $p = \frac{\partial L}{\partial \dot{q}}$. Write down Hamilton's equations relating $\dot{p}$ and $\dot{q}$ to $\frac{\partial H}{\partial q}$ and $\frac{\partial H}{\partial p}$.

A particle of mass m in motion on the x-axis is described by the Lagrangian:

$$L(x, \dot{x}, t) = \tfrac{1}{2}m\dot{x}^2 - \phi(x, t) + \dot{x}A(x, t),$$

where $\phi(x, t)$ and $A(x, t)$ are given functions. Derive the corresponding Hamiltonian. Write down Lagrange's equation and Hamilton's equations for the system, and show that both lead to the following equation of motion:

$$m\ddot{x} = -\frac{\partial \phi}{\partial x} - \frac{\partial A}{\partial t}.$$

If $\phi = \frac{1}{2}m\omega^2 x^2$ and $A = \frac{m}{\Omega}\cos \Omega t$ (where ω, Ω are positive constants, and $\omega \neq \Omega$) find $x(t)$, given initial conditions $x(0) = \dot{x}(0) = 0$.

3. The functions $u(x, y)$ and $v(x, y)$ satisfy the simultaneous partial differential equations

$$\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = x$$
$$-2\frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} - 3\frac{\partial v}{\partial y} = 4x.$$

Show that this system of differential equations is hyperbolic, with characteristics:

$$3x - y = \beta = \text{constant}$$
$$2x + y = \alpha = \text{constant}.$$

Hence show that the Riemann invariants of the system are

$$4u + v - 5x^2 = \text{constant (when } \beta \text{ is constant), and}$$
$$v - u - \tfrac{5}{2}x^2 = \text{constant, (when } \alpha \text{ is constant).}$$

Given that $u(x, 0) = x$ and $v(x, 0) = 0$, find u and v at the point $(x, y) = (1, 3)$.

4. The function $z(x, y)$ satisfies the partial differential equation

$$xy\frac{\partial^2 z}{\partial x^2} + x^2\frac{\partial^2 z}{\partial x \partial y} = y\frac{\partial z}{\partial x}.$$

Show that this equation is hyperbolic, with characteristics

$$\nu = y \quad \text{and} \quad \eta = y^2 - x^2.$$

Hence reduce it to canonical form, and find the general solution.

Find the particular solution such that $z(0, y) = 0$ and $z(x, 0) = x^8$.

5.

(i) Show that the conformal transformation

$$z = c \cosh w$$

where c is a positive real constant and $w = u + iv$, transforms the lines $u =$ constant into confocal ellipses in the z-plane.

(ii) Write down a function $\Phi(u)$ harmonic in the w-plane for $u_1 < u < u_2$ such that $\Phi(u_1) = \Phi_1$ and $\Phi(u_2) = \Phi_2$, where $\Phi_{1,2}$ are constants and $u_1 > 0, u_2 > 0$.

(iii) Two confocal ellipses in the z-plane are maintained at constant potentials: Φ_1 for the inner ellipse and Φ_2 for the outer ellipse. The ellipses have foci at $(\pm c, 0)$ and semi-major axes a_1 and a_2. Show that the solution to Laplace's equation for the potential in the region between the ellipses is given by

$$\Phi = A + Bu(x, y)$$

where A, B are constants you should determine (in terms of $\Phi_{1,2}, a_{1,2}$ and c), and $u(x, y)$ is defined implicitly by the equation:

$$\frac{x^2}{c^2 \cosh^2 u} + \frac{y^2}{c^2 \sinh^2 u} = 1.$$

6. Given that the Fourier Transform of a function $f(x)$ defined on the interval $-\infty < x < \infty$ is

$$F\{f(x);\omega\} = \overline{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx,$$

show that if $f \to 0$ when $x \to \pm\infty$ then

$$F\{\frac{df}{dx};\omega\} = i\omega\overline{f}(\omega).$$

The function $z(x,y)$ satisfies the partial differential equation

$$\frac{\partial^2 z}{\partial y^2} = \frac{\partial^4 z}{\partial x^4}$$

for $y \geq 0$ with boundary conditions as follows:

$$z(x,0) = f(x), \quad z, z_x, \cdots z_{xxx} \to 0 \text{ as } x \to \pm\infty.$$

By using a Fourier Transform, show that $z(x,y)$ satisfies the equation:

$$z(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \overline{f}(\omega)e^{-\omega^2 y}e^{i\omega x}\,d\omega.$$

Then, using the results

$$F\{e^{-a^2x^2};\omega\} = \frac{\sqrt{\pi}}{a}e^{-\frac{\omega^2}{4a^2}},$$

where a is a positive constant, and

$$F^{-1}\{\overline{f}(\omega)\overline{g}(\omega)\} = \int_{-\infty}^{\infty} f(x-\xi)g(\xi)\,d\xi,$$

show that

$$z(x,y) = \frac{1}{2\sqrt{\pi y}}\int_{-\infty}^{\infty} f(x-\xi)e^{-\xi^2/(4y)}\,d\xi.$$

Hence find $z(x,y)$ for the case $f(x) = A\delta(x)$, where A is a constant, and $\delta(x)$ is the Dirac δ-function.

Verify that this solution in fact satisfies the equation

$$\frac{\partial z}{\partial y} = \frac{\partial^2 z}{\partial x^2}.$$

7. Using the Method of Variation of Arbitrary Constants, or otherwise, find the general solutions of the following differential equations:

$$\text{(i)} \qquad \frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \frac{e^{3x}}{x^2}$$

and

$$\text{(ii)} \qquad \frac{d^3y}{dx^3} + \frac{dy}{dx} = \operatorname{cosec} x.$$

[Note that $\int \operatorname{cosec} x\,dx = -\ln(\operatorname{cosec} x + \cot x) + c$.]