**Instructions to candidates**

Full marks can be obtained for complete answers to FIVE questions. Only the best five answers will be taken into account.

**1.** Write down the differential equation satisfied by the function $y(x)$ for which the functional

$$I[y] = \int_a^b F(x, y, y')\, dx \qquad y(a) = y_0, \quad y(b) = y_1$$

is stationary.

Show that if $F$ is not an explicit function of $x$ then the extremal curve satisfies the equation

$$F - y' \frac{\partial F}{\partial y'} = C$$

where $C$ is a constant.

For the case

$$I[y] = \int_0^1 y^2 (1 - y'^2)\, dx \qquad y(0) = 0, \quad y(1) = 1$$

show that the extremal curve satisfies the equation

$$\frac{dy}{dx} = \frac{\sqrt{c^2 - y^2}}{y}$$

where $c$ is a constant. Hence find the extremal curve. Calculate $I[y]$ both for the extremal curve and for the straight line joining the two end points. What can you conclude about the nature of the extremum?

**2.** A dynamical system with one degree of freedom $q(t)$ is described by the Lagrangian $L(q, \dot{q}, t)$, where $\dot{q} \equiv \frac{dq}{dt}$. Write down Lagrange's equation.

The Hamiltonian $H$ is defined as follows:

$$H(q, p, t) = p\dot{q} - L$$

where $p = \frac{\partial L}{\partial \dot{q}}$. Show that, given Lagrange's equation, it follows that Hamilton's equations:

$$\frac{\partial H}{\partial q} = -\dot{p} \quad \text{and} \quad \frac{\partial H}{\partial p} = \dot{q}$$

are also true.

A particle of mass $m$ in motion on the $x$-axis in a potential $V(x)$ is described by the Hamiltonian:

$$H(x, p, t) = \frac{p^2}{2m} e^{-\alpha t} + V(x)$$

where $\alpha$ is a positive constant. Write down Hamilton's equations for the system. Given initial conditions $x(0) = 0$ and $p(0) = p_0$, find $x(t)$ for the two cases

(i)

$$V = V_0, \quad \text{where } V_0 \text{ is constant}$$

(ii)

$$V = kx, \quad \text{where } k \text{ is a constant.}$$

For both cases, describe what happens to the particle as $t \to \infty$.

**3.** The functions $u(x, y)$ and $v(x, y)$ satisfy the simultaneous partial differential equations

$$2\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + 4\frac{\partial v}{\partial y} = 0$$

$$3\frac{\partial u}{\partial x} + 3\frac{\partial v}{\partial x} + \frac{\partial v}{\partial y} = 0.$$

Show that this system of differential equations is hyperbolic, with characteristics:

$$x + y = \beta = \text{constant}$$

$$x + 6y = \alpha = \text{constant}$$

Hence show that the Riemann invariants of the system are $u + 3v = \text{constant}$ and $3u + 4v = \text{constant}$.

Find solutions $u(x, y)$ and $v(x, y)$ such that $u(x, 0) = v(x, 0) = x^2$, and verify that they satisfy the original equations.

**4.** Show that the partial differential equation

$$x^2 \frac{\partial^2 z}{\partial x^2} - y^2 \frac{\partial^2 z}{\partial y^2} = F(x, y, \frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}) \qquad (1)$$

is hyperbolic, with characteristics

$$\nu = xy \quad \text{and} \quad \eta = y/x.$$

Hence show that Equation (1) may be written in canonical form as:

$$4\eta\nu \frac{\partial^2 z}{\partial \eta \partial \nu} - 2\eta \frac{\partial z}{\partial \eta} = -F.$$

Find the general solution of Equation (1) in terms of $x$ and $y$ for the case

$$F = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y} + 4y^2.$$

**5.** (i) Show that the conformal transformation

$$w = i\left(\frac{1-z}{1+z}\right)$$

maps the interior of the unit circle $|z| = 1$ into the upper half $w$-plane, $\operatorname{Im} w > 0$. Indicate in a diagram how the two semicircles $|z| = 1, \operatorname{Im} z > 0$ and $|z| = 1, \operatorname{Im} z < 0$ are mapped into the $w$-plane.

(ii) Show that $\Phi = 1 - \frac{1}{\pi}\phi$, where $w = u + iv = \rho e^{i\phi}$, is a function harmonic in the upper-half $w$-plane, taking the values $\Phi = 0$ for $u < 0, v = 0$ and $\Phi = 1$ for $u > 0, v = 0$.

(iii) Using (i) and (ii), construct a function harmonic inside the unit circle $|z| = 1$, taking prescribed values $F(\theta)$ on its circumference as follows:

$$\begin{aligned} F(\theta) &= 1 \quad \text{for} \quad 0 < \theta < \pi \\ &= 0 \quad \text{for} \quad \pi < \theta < 2\pi. \end{aligned}$$

**6.** Given that the Fourier Transform of a function $f(x)$ suitably defined on the interval $-\infty < x < \infty$ is

$$F\{f(x);\omega\} = \overline{f}(\omega) = \int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx,$$

show that

$$F\{e^{-a^2x^2};\omega\} = \frac{\sqrt{\pi}}{a}e^{-\frac{\omega^2}{4a^2}},$$

where $a$ is a positive constant.

The function $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial u}{\partial t} = c^2\frac{\partial^2 u}{\partial x^2},$$

for $t \geq 0$ and with $c$ constant, and boundary conditions as follows:

$$u(x,0) = g(x), \quad u, u_x \to 0 \text{ as } x \to \pm\infty.$$

By using a Fourier Transform, show that $u(x,t)$ is given by:

$$u(x,t) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} g(x + 2cz\sqrt{t})e^{-z^2}\,dz.$$

Hence find $u(x,t)$ for the case $g(x) = A\delta(x)$, where $A$ is a constant, and $\delta(x)$ is the Dirac $\delta$-function.

**7.** Describe the Method of Variation of Arbitrary Constants for the solution of ordinary linear differential equations.

Using the method, or otherwise, find the general solution of the differential equation

$$\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 3y = \frac{1}{1+e^{-x}}.$$