2MA47

JANUARY 1997

Candidates should answer the whole of Section A and THREE questions from Section B.
Section A carries 55% of the available marks.

SECTION A

1. The utility function U is defined by

$$U(x,y) = (x+3)(y-4), \quad x > 0, y > 4.$$

Find a function $f(x)$ such that $y = f(x)$ is the equation for a typical indifference curve derived from $U(x,y)$. Verify that $f'(x) < 0$ and $f''(x) > 0$ and sketch this curve in commodity space.

[6 marks]

2. Derive geometrically the usual criterion yielding the point of maximum utility $U(x,y)$ subject to a linear budget constraint. Show how to deal with the case where this criterion is not satisfied in (the positive quadrant of) commodity space.

[7 marks]

3. The production function q for a firm is

$$q(x,y) = (xy)^{1/2}$$

where x and y are the quantities used per unit time The total cost of running the firm is

$$C = (x+1)^2 + (y+1)^2$$

Draw a diagram showing several typical isoquant and isocost curves in the input space. Explain briefly the meaning of the expansion path, and indicate it on your diagram. (It is not necessary to calculate the expansion path.)

[7 marks]

4. The cost function of a firm is given by

$$C(q) = q^3 - 5q^2 + 10q + 48$$

Find
(i) the marginal cost function,
(ii) the average total cost function,
(iii) the positive value of q at which the marginal cost equals the average total cost.

[4 marks]

5. The cost function for a firm is

$$C(q) = q^3 - 12q^2 + 93q + 14 .$$

Find the price at which the firm is forced to cease production (in the short term) in an ideal competitive market and derive the supply function of the firm.

[7 marks]

6. Find the price charged by a monopolist with cost function

$$C(q) = q^2 + 5q + 7$$

when the demand function is

$$D(p) = 225 - p$$

where p is the price and q is the amount produced per unit time.

[5 marks]

7. A particular single species population has a low density per capita growth rate of 4 individuals per unit time. There are population equilibria at densities of 0, 2 and 5 individuals per unit measure of habitat.
Given that it is a polynomial, find the simplest possible form of the per capita growth rate as a function of the density x. State with reasons which of the equilibria are stable and which are unstable.

[6 marks]

8. Write down the condition which must be satisfied by the eigenvalues of a 2 x 2 community matrix J at a stable equilibrium point. Derive the stability condition trace J < 0 and det J> 0.

[5 marks]

9. The population densities *x(t), y(t)* of a predator-prey system at time *t* satisfy

$$\frac{\mathrm{d}x}{\mathrm{d}t} = x(5-x) - xy, \quad \frac{\mathrm{d}y}{\mathrm{d}t} = -2y + xy.$$

Which species is the predator and why?
Given that the system has unstable equilibrium points at (0,0) and (5,0) and a stable equilibrium point at (2,3), draw a diagram of the phase plane *OXY*, showing all points of equilibrium, the zero isocline, the infinite isocline, the direction field and a typical trajectory. Show that the line joining the origin to the stable point is an isocline.

[8 marks]

SECTION B

10. A consumer's utility function is

$$U(x, y) = x^2 + 5xy + 4y^2,$$

where x and y are quantities of two commodities X and Y respectively.
Sketch three indifference curves and show that, on such curves

$$dy/dx < 0 \text{ and } d^2y/dx^2 > 0.$$

Hence, show that

$$-5/8 < dy/dx < -2/5.$$

Add the budget line

$$p_1 x + p_2 y = m$$

to your sketch and show that, for $p_1/p_2 < 2/5$, the consumer only buys X and, for $p_1/p_2 > 5/8$, the consumer only buys Y. What proportion of the budget does the consumer spend on X if $p_1/p_2 = 1/2$?

[15 marks]

11. Two companies make the same good. Their cost functions are

$$C_1(q_1) = q_1^2 + 15q_1 + 1$$

and

$$C_2(q_2) = 2q_2^2 + 20q_2 + 1,$$

where q_1 and q_2 are their respective outputs per unit time. The two firms form a duopoly which satisfies an aggregate demand

$$D(p) = 60 - p,$$

where p is the price per unit of the good.
Assuming that each company maximises its profit independently of the other, find both outputs, both profits and the price p.
Suppose that the companies agree to co-operate and maximise their joint profit. Find their new outputs, their new individual profits and the new price p.
Comment very briefly on your results.

[15 marks]

12. Find the equilibrium prices in a perfectly competitive market with supply function

$$S(p) = -p^2 + 11$$

and demand function

$$D(p) = -5p + 17.$$

Determine whether these prices are stable using the static criteria of both Walras and Marshall.
The dynamic behaviour of the price p satisfies, in certain circumstances, the equation

$$dp/dt = k\big(D(p) - S(p)\big),$$

where k is a constant. Justify this equation on Walrasian grounds; what is the sign of k? Obtain the general solution of the equation and show that near the equilibrium prices this solution leads to behaviour which accords with the Walrasian static stability hypothesis.

[15 marks]

13. Two competing species with population densities $x_1(t)$ and $x_2(t)$ at time t are such that

$$\frac{\mathrm{d}x_1}{\mathrm{d}t} = r_1 x_1 (1 - c_{11} x_1 - c_{12} x_2),$$
$$\frac{\mathrm{d}x_2}{\mathrm{d}t} = r_2 x_2 (1 - c_{22} x_2 - c_{21} x_1),$$

where the parameters r_i and c_{ij} are positive.

Find the coexistence equilibrium and determine when it is both feasible (that is at positive densities) and stable.
State Gause's principle.

[15 marks]

14. The respective population densities *x(t)* and *y(t)* of a prey and a predator species may be modelled by the equations

$$dx/dt = x(1-x) - axy,$$
$$dy/dt = sy(1 - y/x),$$

where t is the time and s and a are positive constants. Describe briefly the biological assumptions inherent in this model.
The model has two equilibrium states with prey present. Find these and analyse their stability.
Illustrate your results using a phase plane diagram which includes equilibrium points, zero and infinite isoclines and sketches of typical trajectories.

[15 marks]