Candidates should attempt all questions in Section A and three questions in Section B.

Section A

1. Find the general solution of the differential equation

$$x^2 \frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + 2y = 0.$$

[4 marks]

2. Solve the system of differential equations

$$\dot{x} = x + 4y, \qquad \dot{y} = 2x + 3y,$$

given the initial conditions

$$x(0) = 3, \qquad y(0) = 0.$$

[8 marks]

3. Calculate the coefficients of the Fourier cosine series of the function

$$f(x) = \sin x, \quad 0 \leq x < \pi.$$

Sketch the graph of the cosine series for $-3\pi < x < 3\pi$.

[7 marks]

4. If $\mathcal{L}(f(t)) \equiv \tilde{f}(s)$ denotes the Laplace transform of the function $f(t)$ defined on $[0, \infty)$, show that

$$\frac{d}{ds} \tilde{f}(s) = -\mathcal{L}(t f(t)).$$

[3 marks]

Calculate:

(i) $\mathcal{L}(\sin kt)$,

[3 marks]

(ii) $\mathcal{L}(t \sin kt)$.

[3 marks]

5. The function $u(x,t) = F(x)\sin\omega ct$ is a solution of the equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}.$$

Show that $F(x)$ satisfies the ordinary differential equation

$$F'' + \omega^2 F = 0.$$

[4 marks]

Given that u also satisfies the boundary conditions

$$u(0,t) = u(d,t) = 0,$$

show that the possible values of ω are $n\pi/d$, where n is an integer.

[4 marks]

Sketch $F(x)$ for $n = 2$.

[2 marks]

6. Show that the change of variable

$$\xi = x - y, \ \eta = y,$$

reduces the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} + 2\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 u}{\partial y^2} = 0,$$

to a canonical form.

[4 marks]

Hence find the general solution for $u(x,y)$.

[3 marks]

7. Show that the function

$$u(x,y) = e^x \cos y$$

satisfies the two-dimensional Laplace's equation.

[2 marks]

Write down the Cauchy-Riemann equations involving u and its conjugate harmonic function $v(x,y)$. Find $v(x,y)$.

[6 marks]

Section B

8. The equation

$$\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 20\cos t$$

has the initial conditions: $y(0) = 2, y'(0) = 6$.

(i) Find the solution of this problem without using the Laplace transform.

[7 marks]

(ii) Laplace transform the equation and find the value of $\tilde{y}$, the Laplace transform of y. Hence find the solution, stating explicitly each inverse Laplace transform you use.

[8 marks]

9. A function $u(x, y)$ satisfies Laplace's equation in the rectangle $0 < x < a, 0 < y < b$, together with the homogeneous boundary conditions

$$u(0, y) = u(a, y) = 0, \quad 0 < y < b,$$

on $x = 0$ and $x = a$.

(i) Show that the separable solutions of this boundary value problem are

$$u_n = \sin\frac{n\pi x}{a}(C_n \cosh\frac{n\pi y}{a} + D_n \sinh\frac{n\pi y}{a}),$$

where n is an integer and C_n and D_n are constants.

[8 marks]

(ii) Find the solution to this problem, i.e. find C_n and D_n, given that $u(x, y)$ satisfies the boundary conditions

$$u(x, 0) = 1, \; u(x, b) = 0, \; 0 < x < a,$$

on $y = 0$ and $y = b$.

[7 marks]

10. In an experiment on chemical diffusion a large plane slab of porous material of thickness d has one face, $x = d$, exposed to a volatile gas at a concentration $C = C_1$. The other face, $x = 0$, is exposed to the atmosphere and is at concentration $C = 0$. The concentration $C(x,t)$ of the gas in the slab satisfies the diffusion equation

$$\frac{\partial^2 C}{\partial x^2} = \frac{1}{\kappa}\frac{\partial C}{\partial t}, \quad 0 \leq x \leq d,$$

(where κ is a constant).

Show that the equilibrium concentration is

$$C = C_1 x/d.$$

where C_1 is a constant.

[3 marks]

At time $t = 0$, the concentration over the face $x = d$ is suddenly reduced to zero.

(i) What is the new equilibrium distribution in the slab, (i.e. determine $C(x,t)$ as $t \to \infty$)?

[1 mark]

(ii) Show that the concentration distribution in the slab at times $t > 0$ may be written in the form

$$C(x,t) = \sum_{n=0}^{\infty} b_n \sin\frac{n\pi x}{d} \exp\left(-\frac{n^2\pi^2\kappa t}{d^2}\right),$$

where the b_n are a set of Fourier coefficients.

[7 marks]

(iii) Show that

$$b_n = (-1)^{n+1} 2C_1/n\pi.$$

[4 marks]

11.(i) Writing $\tilde{f}(s)$ for the Laplace transform of $f(t)$, and $H_a(t)$ for the Heaviside (or unit step) function, show that the Laplace transform of $f(t-a)H_a(t)$ is

$$\tilde{f}(s)\exp(-as).$$

[3 marks]

(ii) The function $u(x,t)$ satisfies the partial differential equation

$$\frac{\partial^2 u}{\partial x^2} - 2\frac{\partial^2 u}{\partial x \partial t} + \frac{\partial^2 u}{\partial t^2} = 0, \quad 0 < x < 1,\ t > 0,$$

and the initial and the boundary conditions

$$u(x,0) = 0, \quad \frac{\partial u}{\partial t}(x,0) = 0,\ \frac{\partial u}{\partial x}(x,0) = 0,$$

$$u(0,t) = 0,\ u(1,t) = t.$$

Show that the Laplace transform of $u(x,t)$ with regard to t, denoted by $\tilde{u}$, satisfies the ordinary differential equation

$$\tilde{u}'' - 2s\tilde{u}' + s^2\tilde{u} = 0.$$

[5 marks]

(iii) Find the boundary conditions for $\tilde{u}$ at $x = 0$ and at $x = 1$.

[2 marks]

(iv) Solve this equation for $\tilde{u}$, and hence find the function $u(x,t)$.

[5 marks]

12. The function $u(x,y)$ satisfies the first order partial differential equation

$$y\frac{\partial u}{\partial x} - x\frac{\partial u}{\partial y} = x - y$$

in the domain $x > 0, y > 0$, and the boundary condition $u = x(x-1)$ along $y = 0$. Show that the family of characteristics of this equation are given by

$$x = s\cos t, \quad y = -s\sin t.$$

[8 marks]

Hence determine the function $u(x,y)$.

[7 marks]