



THE UNIVERSITY
of LIVERPOOL

SEPTEMBER 1998 EXAMINATIONS

Degree of Bachelor of Engineering: Year 2

Degree of Master of Engineering: Year 2

Mathematics for Civil Engineers

Time allowed: 3 hours

INSTRUCTIONS TO CANDIDATES

Full marks can be obtained for *six* complete answers. Only the best *six* answers will be taken into account.

A statistical table is attached to the back of this paper.



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1. (a) Find the (unique) solution to

$$\begin{aligned}x + 4y + 4z &= 2, \\3x - 2y - 2z &= -1, \\x + z &= 1,\end{aligned}$$

by using elementary row operations.

- (b) Find the general solution of

$$\begin{aligned}x_1 + 2x_2 + 3x_3 + x_4 &= 1, \\x_1 + 2x_2 + x_3 &= 2, \\-x_1 + 2x_2 + x_3 + 2x_4 &= 4,\end{aligned}$$

by using elementary row operations; express your answer in vector form and interpret it geometrically.

2. (a) Find the (unique) solution to

$$\begin{pmatrix} 2 & 1 & 1 \\ 1 & -1 & 2 \\ -1 & -2 & 2 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -7 \\ 1 \\ 13 \end{pmatrix}$$

by first finding the *adjoint* and *determinant* of the square matrix, and then its inverse.

- (b) Find the solution once more, this time using Cramer's rule.
(c) Check that the solution from Part (a) is the same as the solution from Part (b).



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3. A uniform string is clamped at its ends $x = 0$ and $x = 2l$. Its displacement, $u(x, t)$, at time t is governed by

$$\beta^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}.$$

- (a) Find the most general solution of the form

$$u(x, t) = X(x)T(t)$$

which satisfies the given boundary conditions.

- (b) Given the initial conditions

$$u(x, 0) = I \sin\left(\frac{\pi x}{2l}\right), \quad \frac{\partial u}{\partial t}(x, 0) = J \sin\left(\frac{\pi x}{2l}\right),$$

where I and J are constants, find the particular solution which satisfies these.

4. A mass is attached to a damped spring and is externally forced. Motion is governed by

$$\frac{d^2 y}{dt^2} + 0.3 \frac{dy}{dt} + 13y = f(t),$$

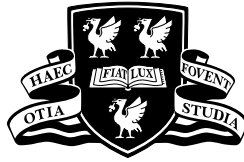
where

$$f(t) = \begin{cases} 1, & \text{in } -T \leq t < 0; \\ 0, & \text{in } 0 \leq t < T \end{cases}$$

and

$$f(t + 2T) = f(t) \quad \forall t.$$

- (a) Find the Fourier series of the forcing function, $f(t)$.
- (b) By using your answer to Part (a) find the motion of the mass (you may neglect the transient part of the solution).



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5. To find maxima or minima of an integral

$$I = \int_{x_1}^{x_2} f(x, y, y_x) dx$$

one usually has to solve the Euler-Lagrange equation,

$$\frac{\partial f}{\partial y} - \frac{d}{dx} \left(\frac{\partial f}{\partial y_x} \right) = 0, \quad x_1 \leq x \leq x_2.$$

(a) Show why, if f does not explicitly depend upon x , one only needs to solve the simpler equation

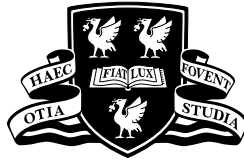
$$y_x \frac{\partial f}{\partial y_x} - f = k,$$

where k is a constant.

(b) Show that the curve $y = y(x)$ in the (x, y) -plane for which the integral

$$\int_0^2 \frac{(1 + y_x^2)^{1/2}}{1 + y} dx$$

is minimised is the arc of a circle.



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6. (a) Obtain the equations which determine the points (x, y) at which the function

$$f(x, y) = 3y(x - 1)^2 + 18y^2(2y - 3) + x^3 - 3x$$

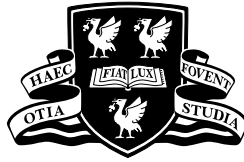
has stationary values. Show that there are four such points, two of which lie on the line $x = 1$, and find their coordinates.

- (b) Show that the function has a minimum at one of the points on $x = 1$ and a saddle point at the other.
- (c) Classify the remaining two stationary points.

7. (a) Compute the partial derivatives $f_x, f_y, f_{xx}, f_{xy}, f_{yy}, f_{xxx}, f_{xxy}, f_{xyy}$ and f_{yyy} of the function

$$f(x, y) = x \ln(x + y).$$

- (b) Use your results from Part (a) to find the Taylor Series at $(0, 1)$ for f up to and including terms cubic in the increments δx and δy .
- (c) Use the approximation to the Taylor Series found in Part (b) to obtain linear, quadratic and cubic approximations for $f(0.2, 0.9)$.



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8. (a) (i) Find a so that

$$f(x) = \begin{cases} ae^{-2\lambda x}, \lambda > 0, & \text{if } x \geq 0, \\ 0 & \text{if } x < 0, \end{cases}$$

is a permissible probability density function (p.d.f.).

- (ii) Sketch the p.d.f. and the cumulative density function (c.d.f.).
(iii) Given $\lambda = 3$, find the probability that x lies in the range $(0.5, 4)$, to 3 s.f.

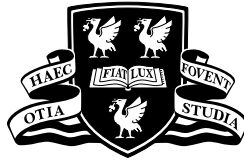
- (b) Experience has shown that on average five cream-teas and three plain-teas are sold by a café in Newton Poppleford during a particular hour of the day.

- (i) Suppose that these events may be satisfactorily modelled by a Poisson distribution,

$$P(x) = e^{-\lambda} \frac{\lambda^x}{x!}, \quad x = 0, 1, 2, \dots$$

Determine the probability density distribution, for (total) teas eaten. State any assumptions you make.

- (ii) What is the probability that less than a total of five teas will be eaten during a particular hour on any day? State your answer to 3 s.f.



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9. An engineering firm produces gizmos, the nominal width of which is 6.20 mm and the tolerance of which is ± 0.05 mm; any gizmos outside the interval (6.15, 6.25) mm are rejected.
- (a) The measured lengths (to the nearest 0.01 mm) of 50 gizmos are given in Table 1. How many gizmos will be *accepted*?
 - (b) Group the data and draw the corresponding histogram.
 - (c) Calculate the mean and standard deviation of the grouped data to 3 s.f.
 - (d) Assume the population is normally distributed with mean and standard deviation equal to the sample values you have calculated. Hence show that approximately 20% of gizmos will be *rejected*.

6.23	6.20	6.25	6.23	6.25
6.21	6.19	6.25	6.20	6.22
6.21	6.27	6.24	6.22	6.23
6.23	6.23	6.23	6.25	6.23
6.22	6.23	6.22	6.21	6.22
6.24	6.20	6.23	6.23	6.24
6.26	6.20	6.22	6.21	6.23
6.24	6.26	6.21	6.24	6.23
6.24	6.22	6.24	6.21	6.26
6.22	6.23	6.25	6.25	6.22

Table 1: Measured length of 50 gizmos, in mm, to the nearest 0.01 mm.

