September 1997 Examinations

Mathematics for Civil Engineers: 2MA2C

Time allowed: 3 hours

INSTRUCTIONS TO CANDIDATES

Use separate answer-books for Sections A and B.

A statistical table is attached to the back of this paper.

Full marks can be obtained for *six* complete answers. Credit will be given for all questions attempted.

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SECTION A

1. (a) Use Gaussian Elimination to show that one of the following systems is inconsistent and the other has an infinite number of solutions (find the rank of $\mathbf{A}|\boldsymbol{b}$, and compare with n, the number of unknowns):

(i)
$$\begin{aligned} -2x_1 + 6x_2 - 4x_3 &= 0 \\ x_1 - 3x_2 + 4x_3 &= -1 \\ 4x_1 - 12x_2 + 4x_3 &= 2 \end{aligned}$$

(ii)
$$\begin{aligned} x_1 - 2x_2 + 2x_3 &= 3 \\ -3x_1 + 6x_2 - 2x_3 &= -9 \\ -4x_1 + 8x_2 + x_3 &= -9 \end{aligned}$$

(b) Find the general solution of whichever of the above systems is consistent and write the solution in parametric form; interpret it geometrically.

2. Calculate the inverse of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & \psi & 0 \\ 0 & 1 & -1 \\ \psi & 0 & 1 \end{pmatrix}$$

by first determining its adjoint; determine the values of ψ for which the matrix is invertible. Using the inverse of $\mathbf{A}$ find the (unique) solution to the system $\mathbf{A}\boldsymbol{x} = \boldsymbol{b}$ given $\boldsymbol{b} = (1, 2, 3)^{\mathrm{T}}$ and $\psi = 2$.

3. (a) Find and classify the stationary points of

$$f(x, y) = a^2x^2 + 2y^2 - \tfrac{1}{2}x^4,$$

when $a \neq 0$.

(b) Comment on the case when $a = 0$. Examine the sign of $f(x, y)$ along both the x-axis and y-axis, when $(x, y) \neq (0, 0)$; hence classify the stationary point(s) in this case.

4. (a) Compute the partial derivatives f_x, f_y, f_{xx}, f_{xy}, f_{yy} of the function

$$f(x,y) = (x^2 + y^2)^{\frac{3}{2}}.$$

(b) Use your results from part (a) to find the Taylor Series at (1,0) for f up to and including terms quadratic in the increments δx and δy.

(c) Use the approximation to the Taylor Series found in part (b) to obtain linear and quadratic approximations for $f(0.9, 0.1)$.

5. (a) Given that the change of variables

$$u = x + 2y, \qquad v = 2x + y,$$

changes $f(x,y)$ to $g(u,v)$, find f_x, f_y, f_{xx}, f_{xy} and f_{yy} in terms of derivatives of g with respect to u and v.

(b) Hence transform the equation

$$5f_{xy} - 2f_{xx} - 2f_{yy} = 0$$

into one for g, and find its general solution.

(c) Find the particular solution which satisfies the boundary conditions

$$f(0,y) = y^2, \qquad f_x(0,y) = 2y.$$

6. The function $f(x)$ is periodic, with period 2, and is defined in the region $-1 < x < 1$ by the formula

$$f(x) = \begin{cases} 1 - x, & \text{if } -1 < x < 0; \\ 0, & \text{if } 0 < x < 1. \end{cases}$$

(a) Sketch the graph of $f(x)$ for $-2 < x < 2$.

(b) Find the Fourier series for this function.

(c) Write out the series explicitly up to and including terms in $\cos 5\pi x$ and $\sin 5\pi x$.

7. (a) Show that the length of a small element of a curve is approximately given by

$$\sqrt{1 + (\delta y/\delta x)^2}\ \delta x.$$

(b) Deduce that the area of a surface, S, generated by rotating the plane curve joining the points (x_1, y_1) and (x_2, y_2) about the x-axis, is given by

$$S = \int_{x_1}^{x_2} 2\pi y \sqrt{1 + (y')^2}\ \mathrm{d}x.$$

(c) Find the curve $y(x)$ which generates the smallest value for S. (You may assume that, for the given end points, solving the Euler-Lagrange equation will yield this.)

HINTS:

(i) The Euler-Lagrange equation is

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right) = 0.$$

(ii) $f_x = 0$ in the above case and

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(y'\frac{\partial f}{\partial y'} - f\right) = -y'\left[\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y'}\right)\right] - \frac{\partial f}{\partial x}.$$