January 1997 Examinations

Mathematics for Civil Engineers: 2MA2C

Time allowed: 3 hours

INSTRUCTIONS TO CANDIDATES

Use separate answer-books for Sections A and B.

A statistical table is attached to the back of this paper.

Full marks can be obtained for *six* complete answers. Credit will be given for all questions attempted.

SECTION A

1. (a) Show that the equations

$$\begin{aligned} 3x - y + 2z &= 2 \\ 2x + y + 3z &= 3 \\ kx + 3y + 4z &= l \end{aligned}$$

do *not* have a unique solution when $k = 1$.

(b) With $k = 1$, for which values of l do the equations have:

(i) an infinite family of solutions,

(ii) no solution.

(c) Solve the equations when $k = 1$ and $l = 4$ writing, your answer in parametric, vector form. Interpret the solution geometrically.

2. (a) Find the inverse, $\mathbf{A}^{-1}$, of the matrix

$$\mathbf{A} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 1 & 5 & 12 \end{pmatrix}.$$

You should check your answer by showing that $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, where $\mathbf{I}$ is the 3×3 identity matrix.

(b) Using your result from part (a), find the (unique) solution to the equations

$$\begin{aligned} x + 2y + 3z &= 4, \\ x + 3y + 5z &= 2, \\ x + 5y + 12z &= 7. \end{aligned}$$

3. (a) Find and classify the stationary points of

$$f(x, y) = x^2 + a^2y^2 - \tfrac{1}{2}y^4,$$

when $a \neq 0$.

(b) Comment on the case when $a = 0$. Examine the sign of $f(x, y)$ along both the x-axis and y-axis, when $(x, y) \neq (0, 0)$; hence classify the stationary point(s) in this case.

4. (a) Compute the partial derivatives f_x, f_y, f_{xx}, f_{xy}, f_{yy}, f_{xxx}, f_{xxy}, f_{xyy} and f_{yyy} of the function

$$f(x, y) = (x^2 + y^2)^{\frac{1}{2}}.$$

(b) Use your results from part (a) to find the Taylor Series at (1,0) for f up to and including terms cubic in the increments δx and δy.

(c) Use the approximation to the Taylor Series found in part (b) to obtain linear, quadratic and cubic approximations for $f(0.9, 0.1)$.

5. (a) Find values of the numbers a and b such that the change of variables

$$\xi = x + ay, \qquad \eta = x + by$$

transforms the equation

$$\frac{\partial^2 u}{\partial x^2} + 5\frac{\partial^2 u}{\partial x \partial y} + 6\frac{\partial^2 u}{\partial y^2} = 0$$

into

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0.$$

(You should use a change of variables in which $a \neq b$.) Hence show that the general solution of the equation is given by

$$u(x, y) = f(3x - y) + g(2x - y).$$

(b) Using the result from part (a) find the solution of the above differential equation which satisfies the conditions

$$u(0, y) = y^3, \qquad u_x(0, y) = y^2.$$

6. A function $g(x)$ is defined by

$$g(x) = x^2 \quad \text{for} \quad -L \le x < L \quad \text{and} \quad g(x+2L) = g(x).$$

(a) Sketch the graph of g for $-4L \le x \le 4L$.

(b) Find the Fourier Series of g. You may use the results

$$\int x^m \sin ax \, \mathrm{d}x = -\frac{1}{a} x^m \cos ax + \frac{m}{a} \int x^{m-1} \cos ax \, \mathrm{d}x,$$

$$\int x^m \cos ax \, \mathrm{d}x = \frac{1}{a} x^m \sin ax - \frac{m}{a} \int x^{m-1} \sin ax \, \mathrm{d}x.$$

Write out this series explicitly up to terms in $\cos\left(\frac{5\pi x}{L}\right)$ and $\sin\left(\frac{5\pi x}{L}\right)$.

7. A mine shaft is to driven from (x_1, y_1) to (x_2, y_2) through strata of varying composition. Digging costs per unit length are given by a continuous function, $c(y)$. All other expenses for the project are given by the cost function

$$C = 100 + 20 \int_{x_1}^{x_2} xy(xy_x + y)\,\mathrm{d}x.$$

(a) Explain briefly why the overall expenses corresponding to a curve $y(x)$ are

$$E = C + D = 100 + \int_{x_1}^{x_2} \left\{ 20xy(xy_x + y) + c(y)\sqrt{1 + y_x^2} \right\} \mathrm{d}x.$$

Hint: first show that the total cost of digging from (x_1, y_1) *to* (x_2, y_2) *is given by*

$$D = \int_{x_1}^{x_2} c(y)\sqrt{1 + y_x^2}\,\mathrm{d}x.$$

(b) The Euler-Lagrange equation is

$$\frac{\partial f}{\partial y} - \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\partial f}{\partial y_x}\right) = 0.$$

(i) If f does not explicitly depend upon x the integration of the Euler-Lagrange equation is greatly simplified. Explain how. (You do *not* need to prove that your answer is correct.)

(ii) If a term in f is the total derivative with respect to x of some function then integration of the equation is again simplified. Explain how. (You do *not* need to prove that your answer is correct.)

(c) Noting that $xy(xy' + y) = (\mathrm{d}/\mathrm{d}x)(\frac{1}{2}x^2y^2)$, and using your answers to (i) and (ii), show that the function $y(x)$ which minimises E is given by

$$x = \pm k \int \frac{\mathrm{d}y}{\sqrt{c^2(y) - k^2}}.$$

SECTION B

To be filled in by Dr Al-Bayati...