Instructions to candidates

Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55% of the available marks.

Take $g = 9.81$ m s^{-2}. Give numerical answers to 3 significant figures.

SECTION A

1. Water enters a bath through a tap and leaves through a plug hole. When the tap is on and the plug is in the plug hole the height, h, of the water in the bath increases at the rate of 2 cm per minute. When the tap is off and the plug is out, the height decreases at the rate of 1 cm per minute.

Write down the differential equation for h in the case when the plug is out and the tap is on. Solve your equation to find the height of the water in the bath 10 minutes after it was 5 cm. [6 marks]

2. The rate of decrease in temperature of a cup of coffee is proportional to its temperature difference with the surroundings at 20°C. Given that the constant of proportionality is 0.15°C min^{-1}, show that

$$\frac{\mathrm{d}T}{\mathrm{d}t} + 0.15T = 3,$$

where T is the temperature at time t.

Solve this equation to find the temperature of the coffee after 5 minutes, given that its initial temperature was 90°C. [7 marks]

3. John has a savings account at a Building Society. On the last day of each year, interest at 5% is paid on the balance (B) at 9:00 on that day. At 2pm, John withdraws, in cash, 20% of B from his account and deposits a cheque for £300. Show that at the end of year $(m + 1)$ his balance $n(m + 1)$ is given by

$$n(m + 1) = 0.85n(m) + 300.$$

Show that the equilibrium solution of this equation is £2000. By considering what happens to £2500 and £1500 over a period of 3 years, find whether the equilibrium is stable or not.

This information can be obtained directly from one of the numbers in the given equation. Which one, and why?

[8 marks]

4. A lorry driver does either short or long journeys. If he drives a long journey one day the probability that he drives a short one the next is 0.7 and vice versa.

Show that the probability of a long journey on day t, $P(long, t)$, is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}P(long, t) = 0.7 - 1.4P(long, t).$$

Given that $P(long, 0) = 0$, solve this equation and show that $P(long, t)$ never exceeds 0.5. [7 marks]

5. At time t, a particle has acceleration

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = 2\sin(2t)\mathbf{i} + 2\cos(2t)\mathbf{j} \text{ m s}^{-2}\ .$$

At time $t = 0$ s, it is at the origin, with $\mathbf{v} = \mathbf{0}$ m s^{-1}. Find its position vector at time t.

Show that the point executes circular motion about $(t\mathbf{i} + \frac{1}{2}\mathbf{j})$ and hence deduce that the particle moves like a point on a moving bicycle wheel of radius a, where a should be given. [8 marks]

6. A ship of mass m kg, travelling in a straight line, applies a constant braking force of $\frac{1}{4}m$ N. The ship experiences a resistive force of $4mv^2$ N, where v m s^{-1} is its speed. Write down Newton's equation of motion.

Given that the initial speed is u m s^{-1}, show that it will come to rest after a time T, where T is given by

$$T = \frac{1}{4}\int_0^u \frac{\mathrm{d}v}{\frac{1}{16} + v^2}.$$

Evaluate this integral (in terms of u). [6 marks]

7. The equation for the displacement, z, of a forced harmonic oscillator is

$$\ddot{z} + 25z = 18\sin 4t.$$

At time $t = 0$, $z = 0$ and $\dot{z} = 3$. Find z at time t. [7 marks]

8. A ball is thrown directly from the origin with a speed of u m s^{-1} at an angle of 45° to the horizontal. Show that the equation of the path taken by the ball is given by

$$y = x - \frac{g}{u^2}x^2.$$

[6 marks]

SECTION B

9. For the process $A \to B \to C$, the amounts of A, B and C at time t are given by a, b and c respectively, which satisfy the differential equations

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -2a, \quad \frac{\mathrm{d}b}{\mathrm{d}t} = 2a - 3b, \quad \frac{\mathrm{d}c}{\mathrm{d}t} = 3b.$$

Show that $a + b + c$ is conserved.

Initially, $a = 1$, $b = 0$ and $c = 0$. Solve these equations in turn to show that

$$c = 1 - 3\mathrm{e}^{-2t} + 2\mathrm{e}^{-3t}.$$

Check that $a + b + c$ is conserved. [15 marks]

10. An animal population Y is the principal predator of another animal population X. The interaction of the two populations is modelled by the coupled differential equations

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{10}x \qquad \frac{\mathrm{d}x}{\mathrm{d}t} = -\frac{1}{10}y,$$

where $x(t)$ and $y(t)$ represent the numbers in populations X and Y respectively, as functions of time t, measured in years.

Show that if the variables are written $x(t) = a\exp(\lambda t)$ and $y(t) = b\exp(\lambda t)$, the eigenvalues are given by

$$\lambda = \pm 0.1\sqrt{-1}.$$

[4 marks]

Thus, writing $x(t)$ as

$$x(t) = A\cos(0.1t) + B\sin(0.1t),$$

obtain a similar expression for $y(t)$ involving the same constants.

Given that initially there are 10^4 of X and 10^2 of Y, determine the constants A and B.

Find the time it takes for X to become extinct. [7 marks]

From your results, or otherwise, draw a phase diagram for this situation. [4 marks]

11. The number $n(t)$, in units of 10^4, of trout in a trout farm satisfy the differential equation, when $n > 0$,

$$\frac{\mathrm{d}n}{\mathrm{d}t} = 6n - \frac{1}{2}n^2 - f$$

where f is the effect of fishing.

If there is no fishing, what is the equilibrium number of trout? [3 marks]

However, there is fishing. Given that $f = \frac{11}{2}$, sketch the graph of $\mathrm{d}n/\mathrm{d}t$ against n and on this graph indicate how $n(t)$ behaves for various values of $n(0)$. [9 marks]

Suggest why f should not exceed 18. [3 marks]

12. Show that, for $a < b$, $(b-x)(x-a)$ has a maximum at $x = (a+b)/2$.

A light elastic spring of length L has modulus $\lambda = 2mg$. It lies on a smooth horizontal table with one end fixed at the origin. A particle of mass m is fixed to the other end which can move along the $x-$axis. Derive an expression for the energy stored in the spring when the particle is at the point x. [4 marks]

The particle is held at rest with $x = \frac{3}{2}L$ and released at time $t = 0$. By considering conservation of energy, or otherwise, show that

$$\dot{x}^2 = \frac{2g}{L}\left[\left(\frac{3L}{2} - x\right)\left(x - \frac{L}{2}\right)\right].$$

[4 marks]

A similar system is suspended vertically from a fixed point O on a ceiling. The $y-$axis is vertically downwards. When the system is at rest show that

$$y = \frac{3}{2}L.$$

The particle is pulled down a further distance $\frac{1}{2}L$ and then released from rest. Show that

$$\dot{y}^2 = \frac{2g}{L}[(2L - y)(y - L)].$$

Hence show that the maximum speed of the particle has the same value for the horizontal and for the vertical motion.

[7 marks]

13. The position vector for a particle moving in a plane perpendicular to $\mathbf{k}$ is $\mathbf{r} = r\hat{\mathbf{r}}$, where $\hat{\mathbf{r}} = \cos\theta\mathbf{i} + \sin\theta\mathbf{j}$.

Given that $\underline{\hat{\theta}}$ is defined by

$$\underline{\hat{\theta}} = \frac{\mathrm{d}}{\mathrm{d}\theta}\hat{\mathbf{r}}$$

express $\underline{\hat{\theta}}$ in terms of $\cos\theta$, $\sin\theta$, $\mathbf{i}$ and $\mathbf{j}$. Hence show that

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{\mathbf{r}} = \frac{\mathrm{d}\theta}{\mathrm{d}t}\underline{\hat{\theta}}.$$

Show that

$$\hat{\mathbf{r}} \times \underline{\hat{\theta}} = \mathbf{k}.$$

[4 marks]

Show that its velocity is given by

$$\mathbf{v} = \dot{r}\hat{\mathbf{r}} + r\dot{\theta}\underline{\hat{\theta}}$$

and its acceleration by

$$\frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = (\ddot{r} - r\dot{\theta}^2)\hat{\mathbf{r}} + \frac{1}{r}(2r\dot{r}\dot{\theta} + r^2\ddot{\theta})\underline{\hat{\theta}}.$$

[7 marks]

Show, from your results, that for a central force, $mr^2\dot{\theta}$ is a constant and, hence, that the angular momentum $\mathbf{L} = m\mathbf{r} \times \mathbf{v}$ is a constant of the motion.

[4 marks]