Candidates should attempt all questions in Section A and three questions in Section B.

Section A

1. In the rush hour 3 people join the queue for a bus every minute. At time $t = 0$, (where t is in minutes) there are no people in the queue. The buses come every 5 minutes, starting at $t = 5$ and eleven people are allowed on the bus. Let $n(t)$ represent the number of people in the queue at time t, and suppose it takes all real values. Assuming people join the queue at a constant rate, draw a graph of $n(t)$ for $t \leq 15$. What is the value of t at which $n(t)$ has the value 16 for the first time?

[7 marks]

2. The temperature $T(t)$°C in a room being heated by an electric fire approximately obeys the equation:

$$dT/dt = 4.5 - T/5,$$

where the second term arises because of heat loss. Find the temperature at time t given that initially $T = 12.5$°C and find the final temperature.

[7 marks]

3. A telephone kiosk can either be "in use" or "empty". If at time t minutes it is either "in use" or "empty", the probability per minute that it is "empty" is 0.4 . Assume that this is a two-state process with $P(U, t)$ as the probability that the kiosk will be in use at time t, show that

$$\frac{dP(U,t)}{dt} = 0.6 - P(U,t).$$

Solve this equation, given that at time $t = 0, P(U, 0) = 1$. What is the long-term value of $P(U, t)$?

[8 marks]

4. A particle with position vector $\mathbf{r}(t)$ at time t, has velocity

$$2\mathbf{i}e^{-t} - 16(5\mathbf{j} + \mathbf{k})e^{-4t}.$$

The particle is initially at the origin. Find its position vector at time $t > 0$. Show also that the particle travels in a plane.

[8 marks]

5. A boat of mass m Kg travels in a straight line and its engine exerts a force T Newtons against a resisting force Tv/u Newtons, where v m/s is the velocity and u is a constant. Write down the equation of motion for the boat. Its initial velocity is 0 m/s. Show that its velocity at time t is

$$u(1 - e^{-Tt/mu}).$$

How far does it travel in that time?

[8 marks]

6. The equation for the displacement, x, of a damped harmonic oscillator is

$$\ddot{x} + 6\dot{x} + 25x = 0.$$

At time $t = 0$, $x = 1$ and $\dot{x} = -3$. Find the value of x at time t.

[8 marks]

7. A ball is thrown with a speed of 20 m/s, at an angle θ to the horizontal. Write down or derive the equations for the horizontal and vertical distances, x and y respectively, travelled by the ball in t seconds.

[3 marks]

It is supposed to hit an object at a point 18m away horizontally and 1.5m above the point of projection. Find the possible value(s) of θ, leaving your result in the form of an inverse trigonometric function.
(you can take g to be 10m/s^2. Hint: $\sec^2\theta = 1 + \tan^2\theta$.)

[6 marks]

Section B

8. The attraction of a ride at a themepark depends on the number, $n(t)$, of people in the queue at time t minutes. Taking $n(t)$ as a continuous variable, it is estimated that the rate at which people join the queue is $24 + 10n - n^2$, per minute.
(i) If the ride is temporarily not working, what is the equilibrium number of people in the queue?

[4 marks]

(ii) When the ride is working, the rate at which the ride can take people from the queue is a per minute. When $a = 33$, what are the possible equilibrium values of $n(t)$ and which ones are stable?

[8 marks]

(iii)What happens if $a > 49$? (In particular say how many people per minute take the ride.)

[3 marks]

9. A model for the number $x(t)$ of rat fleas per unit area at time t uses the equation

$$dx/dt = xy,$$

where $y(t)$ is the number of rats per unit area at time t . The fleas make the rats ill. Thus the number of rats is thought to satisfy the equation:

$$dy/dt = y^2(5 - x).$$

Find the equation for dy/dx and integrate it, given that initially $x = 1.0$ and $y = 50$.

[9 marks]

Sketch the graph of y against x, indicating the direction that x and y change with time. Describe what happens to the two populations.

[6 marks]

10. A car of mass m Kg is travelling at u m/s along a straight road when, at time $t = 0$, it begins to lose power. The force that the engine provides can thus be written as $T - at$ Newtons for $t < T/a$, and 0 otherwise. The frictional force is T Newtons, when the car is moving; if the car stops, it remains stationary.

Write down Newton's equation of motion for $t < T/a$, and for $t > T/a$.

[4 marks]

Find how long it takes for the car to stop when:

(i) $2mua < T^2$?

[4 marks]

(ii) $2mua > T^2$?

[7 marks]

11. An aeroplane of mass m Kg, when it lands, decelerates by two mechanisms: firstly at a constant rate by using its engine as a brake of magnitude md Newtons and secondly by increasing its air resistance by an amount mkv Newtons where k and d are constants and v is the velocity of the aeroplane on the runway. Its initial velocity on the runway is um/s. Show that it takes a time $\ln(1 + ku/d)/k$ seconds for the aeroplane to come to rest.

[7 marks]

Show that $\frac{dv}{dt} = v\frac{dv}{ds}$ where s is the distance the aeroplane has travelled along the runway after t seconds. Hence or otherwise find the distance the aeroplane travels along the runway before it comes to rest.

(Hint: $\frac{v}{d+kv} = [1 - \frac{d}{d+kv}]/k$.)

[8 marks]