Candidates should attempt all questions in Section A and three questions in Section B.

Section A

1.(i) 3 people join the queue for a ride at Euro-Disney every two minutes and 5 people leave the queue every four minutes to take the ride. Write down a differential equation for $n(t)$, the number of people in the queue at time t, where the unit of time is one minute. How long does it take for there to be 20 people in the queue, given that initially there were none?

[4 marks]

(ii) A better model takes into account the fact that the rate of arrival of people at the queue is reduced, estimated to be by a factor $(1 - n/180)$, by people not wanting to queue a long time. Write down the new differential equation and, with the same initial condition as above, solve it. What is the maximum possible number of people in the queue?

[8 marks]

2. In a model, a theoretical biologist relates the number $N(m+1)$ of a species of beetle in year $(m+1)$ to the number $N(m)$ in year m by the equation:

$$N(m+1) = (1+b)N(m) - c,$$

where the term c is due to predation. b, c are positive constants. Find the equilibrium value of N and the value of $N(m)$ in terms of $N(0)$, the value of $N(m)$ in year 0.

[6 marks]

3. On average a female salmon mates with 5 male salmon in a season. Assuming a Poisson distribution for the probability of mating with a number of males, find to 3 significant figures the probability of a female salmon mating with :(i) one male salmon: (ii) five male salmon.

[5 marks]

4. A particle with position vector $\mathbf{r}(t)$ at time t, has velocity

$$\mathbf{i}e^{-t} - 10(3\mathbf{j} + \mathbf{k})e^{-5t}.$$

The particle is initially at the origin. Find its position vector at time $t > 0$. At what position does the particle come to rest? Show also that the particle travels in a plane.

[7 marks]

5. A boat of mass m Kg travels in a straight line and its engine exerts a force T Newtons against a resisting force Tv/u Newtons, where v m/s is the velocity and u is a constant. Write down the equation of motion for the boat. Its initial velocity is 0 m/s. Show that its velocity at time t is

$$u(1 - e^{-Tt/mu}).$$

How far does it travel in that time?

[8 marks]

6. A pendulum obeys the equation

$$\ddot{\theta} + 10\dot{\theta} + 169\theta = 0,$$

where θ is the angle of the pendulum to the vertical. At time $t = 0$, $\theta = \pi/12$ and $\dot{\theta} = 0$. Find the value of θ at time t.

[8 marks]

7. A ball is thrown with a speed of 10 m/s, at an angle θ to the horizontal. Write down or derive the equations for the horizontal and vertical distances, x and y respectively, travelled by the ball in t seconds.

[3 marks]

It is supposed to hit an object at a point 9m away horizontally and 0.5m above the point of projection. Find the possible value(s) of θ, leaving your result in the form of an inverse trigonometric function.
(you can take g to be 10m/s^2. Hint: $\sec^2\theta = 1 + \tan^2\theta$.)

[6 marks]

Section B

8. The number $n(t)$, in units of 10^6, of mackerel in the seas around Britain approximately obeys the equation, for $n \neq 0$:

$$dn/dt = 6n - n^2/2 - f,$$

where f is a constant representing the fishing quota per year allowed by the E.U. For $n = 0, dn/dt = 0$. The unit of time is one year. After consultation the E.U. decide that a suitable value of f is 11/2.

What are the equilibrium values of n? Sketch the graph of dn/dt against n, and hence describe the way that $n(t)$ varies for various initial populations $n(0)$.

[9 marks]

What is the long term behaviour of $n(t)$?

[2 marks]

Outline the steps you would take to solve the above differential equation. 1Indicate how you would use the exact solution to verify your conclusions above.

[4 marks]

9. A model for the number $x(t)$ of rat fleas per unit area at time t uses the equation

$$dx/dt = x^2 y,$$

where $y(t)$ is the number of rats per unit area at time t . The fleas give the rats a lethal disease. Thus the number of rats is thought to satisfy the equation:

$$dy/dt = y - x^2 y.$$

Find the equation for dy/dx and integrate it, given that initially $x = 3/4$ and $y = 5/12$.

[9 marks]

Sketch the graph of y against x, indicating the realistic part of the graph and the direction that x and y change with time. Describe what happens to the two populations.

[6 marks]

10. (i) A car of mass m Kg is travelling at u m/s along a straight road when, at time $t = 0$, it begins to lose power. The force that the engine provides can thus be written as $T - at$ Newtons for $t < T/a$, and 0 otherwise. The friction is T Newtons, when the car is moving.

Write down Newton's equation of motion for $t < T/a$, and for $t > T/a$.

[4 marks]

Find how long it takes for the car to stop when:

(i) $2mua < T^2$?:

[4 marks]

(ii) $2mua > T^2$?

[7 marks]

11. A bungee jumper of mass m Kg falls under the force of gravity. The unstretched rope has length l m, which means that he has free fall for this distance. What is his velocity after falling this far?

[3 marks]

After this the rope exerts a force $-mb^2(x - l)$ Newtons, where x is the total distance the jumper falls. Show that

$$v^2 = 2gx - (bx)^2 + 2b^2lx - (bl)^2.$$

where v is the velocity in m/s.

[8 marks]

What is the maximum distance the jumper falls?

[4 marks]