Instructions to candidates

Candidates should answer the WHOLE of Section A and THREE questions from Section B. Section A carries 55% of the available marks.

Take $g = 9.80$ m s^{-2}. Give numerical answers to 3 significant figures.

SECTION A

1. A rat population has a growth rate of 5% per month. As the result of an extermination programme 40 rats per month are killed. If the initial population is 600, how many months does it take for all rats to be exterminated?

[6 marks]

2. The differential equation for a two-state stochastic model of the weather at a particular place is given by

$$\frac{\mathrm{d}}{\mathrm{d}t}P(wet,t) = \frac{1}{6}P(dry,t) - \frac{2}{3}P(wet,t)$$

where $P(wet,t)$ is the probability that it is wet on day t.

What do the $\frac{1}{6}$ and the $\frac{2}{3}$ represent?

Given that it is *dry* on day $t = 0$, find $P(wet,t)$.

Show that $P(wet,t) \to \frac{1}{5}$ as $t \to \infty$.

[8 marks]

3. Water enters a uniform cylindrical container with a cross-sectional area of 1 cm^2 at the constant rate of 10 $\text{cm}^3\ \text{s}^{-1}$. There is a hole in the bottom of the container and water leaks out at a rate proportional to the height, h, of the water in the container at time t. Show that the differential equation for h is

$$\frac{\mathrm{d}h}{\mathrm{d}t} = 10 - kh.$$

Given that the height of the water approaches a maximum value of 20 cm, solve your differential equation to find how long it takes for the water to reach a height of 10 cm if the container is initially empty.

[6 marks]

4. Let $x(t)$ represent the number of rabbits and $y(t)$ the number of foxes in a particular population at time t. In a simplified model, these satisfy the differential equations:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 100 - y$$

$$\frac{\mathrm{d}y}{\mathrm{d}t} = x - 30.$$

At time $t = 0$, $x = y = 60$. Show that the point (x, y) lies on a circle, whose centre and radius should be stated.

[6 marks]

5. At time t, a particle has velocity

$$\mathbf{v}(t) = (5\mathbf{i} - 3\mathbf{j})\sin(2t) + 2\mathbf{k}\cos(3t) \text{ m s}^{-1} .$$

At time $t = \frac{1}{2}\pi$ s, it passes through the origin. Find its position vector at any later time. Show that the particle's motion is confined to a plane.

[8 marks]

6. The engine of a speedboat of mass m kg exerts a constant force mku N, where k and u are constants. When the boat is travelling in a straight line with speed v m s^{-1}, it experiences a resistive force of mkv N. Its initial speed is 0 m s^{-1}. Write down Newton's equation of motion. Show that,

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathrm{e}^{kt}v\right] = k\mathrm{e}^{kt}u.$$

Hence, or otherwise, find the speed of the speedboat at time t.

[7 marks]

7. The equation for the displacement, x, of a damped harmonic oscillator is

$$\ddot{x} + 8\dot{x} + 25x = 0.$$

At time $t = 0$, $x = 0$ and $\dot{x} = 6$. Find x at time t and sketch its graph.

[7 marks]

8. A ball is thrown directly towards a wall with a speed of 20 m s^{-1} at an angle of 60^o to the horizontal. The wall is 20 m from the point of projection and is of height 16 m. Given that the ball is thrown from a height of 2 m, find whether it hits or passes over the wall.

[7 marks]

SECTION B

9. A radioactive substance X decays to a radioactive substance Y which then decays to the stable substance Z. X decays at a rate proportional to the mass of X present at time t, with the constant of proportionality equal to 0.02 per year. Initially the mass of X is 2 kg and there is no Y or Z.

Let x be the mass of X at time t. Write down the differential equation satisfied by x. Solve this equation to find the mass of X at time t.

The mass, y, of Y increases because of the decay of X but decreases at a rate proportional to the mass of Y present at time t, with the constant of proportionality equal to 0.01 per year.

Write down the differential equation satisfied by y. Solve this equation to find the mass of Y at time t.

[9 marks]

Write down and solve the differential equation to find the mass of Z at time t.

What is the mass of Z after one hundred years? Give your answer to three significant figures.

[6 marks]

10. The rate at which people join a queue is $88 + 42n - n^2$ per minute, where $n(t)$ is the number of people in the queue.

If no one is leaving the queue, what would be the final number of people in the queue?

However, people are leaving the queue at the rate of 168 per minute. Show that $n(t)$ satisfies the differential equation

$$\frac{\mathrm{d}n}{\mathrm{d}t} = -(n^2 - 42n + 80).$$

What are the possible equilibrium values of $n(t)$ and which ones are stable?

[9 marks]

Solve the differential equation to find $n(t)$ given that there are 20 people on the queue at time $t = 0$. [Hint: use partial fractions.]

[6 marks]

11. The position vector of a particle of mass m at time t is given by

$$\mathbf{r} = 2a\cos(\omega t)\mathbf{i} + 2a\sin(\omega t)\mathbf{j}.$$

Describe, fully, its motion.

Find its velocity and acceleration at time t. Justify the expression "acceleration at constant speed".

[8 marks]

Show that the force which produces the motion of this particle is a central force. If this force is produced by an elastic string of natural length a and modulus λ which has the other end fixed at the origin, show that

$$\omega^2 = \frac{\lambda}{2ma}.$$

[5 marks]

Another particle has the position vector

$$\mathbf{r} = 2a\cos(\omega t)\mathbf{i} + 2a\sin(\omega t)\mathbf{j} + a\omega t\mathbf{k}.$$

Briefly describe its motion.

[2 marks]

12. Show that if $v = \frac{\mathrm{d}s}{\mathrm{d}t}$, then

$$\frac{\mathrm{d}v}{\mathrm{d}t} = v\frac{\mathrm{d}v}{\mathrm{d}s}.$$

[1 marks]

A particle of mass m kg is thrown vertically upwards with an initial velocity u m.s^{-1}. It experiences an air resistance of mkv N, when its velocity is v m.s^{-1}, where k is a constant.

What are the units of k?

Show that it reaches a height

$$h = \frac{u}{k} - \frac{g}{k^2}\ln\left(1 + \frac{ku}{g}\right).$$

[9 marks]

Given that it takes a time T to reach this height, show that

$$\int_u^0 \frac{\mathrm{d}v}{g + kv} = -\int_0^T \mathrm{d}t.$$

Hence, or otherwise, find T and deduce that

$$hk + gT = u.$$

[5 marks]

13. Briefly explain the Conservation of Energy for a particle moving in one dimension. [1 marks]

A bungee jumper of mass 70 kg, initially at rest, drops from a bridge 60 m above a river and then falls under the force of gravity. The unstretched rope has length 20 m. What is her velocity after she has fallen this distance, i.e. immediately before the rope starts to stretch?

The rope then exerts a force $-14g(x - 20)$ N, where x is the distance downwards from the bridge. Show that at this point, her speed is given by

$$v^2 = 2g\left(x - \frac{1}{10}(x - 20)^2\right).$$

[5 marks]

What is the maximum distance that she falls? [5 marks]

For the above we have neglected friction due to air resistance. If friction were to be considered the bungee jumper would oscillate a few times and then come to rest. How far would she then be from the bridge?

[4 marks]