Candidates should attempt all questions in Section A and three questions in Section B.

Section A

1. A mathematical model for the number $n(t)$ of people in the queue at a ride at Euro Disney is based on measurements which showed that approximately $(3 - n/50)$ people join the queue every two minutes while 5 people left the queue every four minutes. Write down the differential equation for $n(t)$ and solve it, given that initially the queue is empty. How many people are there in the queue in the long term?

[7 marks]

2.(i) A road runs North-South (N-S) and has a junction, with traffic-lights, with a road that runs East-West (E-W). The amount of traffic is the same in both directions on the N-S road, as it is on the E-W road. On the N-S road, traffic travelling south reaches the junction at the rate of 10 cars per minute. On each cycle of the traffic-lights, the lights for the N-S road are red for $(t_2 + 0.5)$ minutes, where t_2 minutes is the time the lights are green on the E-W road. How many cars facing south are at the junction at the end of the period when the lights are red, given that there were no such cars at the beginning of this period?

[2 marks]

(ii) On the E-W road 6 cars per minute are travelling in each direction. The lights on the N-S road are green for t_1 minutes per cycle so that they are red for the E-W road for $(t_1 + 0.5)$ minutes per cycle. Thus the cycle is $(t_1 + t_2 + 0.5)$ minutes long, the 0.5 minutes being for safety purposes. When the appropriate lights are green the cars leave the junction (on either road , in either direction) at the rate of 18 cars a minute. Show that

$$8t_1 - 10t_2 - 5 \geq 0$$

if there are no queues of cars left on the N-S road at the junction at the end of the time when the lights were green. Derive a similar inequality to ensure that there are no queues of cars left on the E-W road. Show that to satisfy these inequalities $t_2 \geq 1.5$ minutes; derive a similar condition for t_1.

[7 marks]

3. A disease infects people in such a way that if they feel ill on one day, the probability that they feel ill the next is 0.7 . If however they feel well on one day the probability that they feel well the next day is 0.6 . Consider this as a two-state process and write down the equation for $\frac{dP(W,t)}{dt}$ in terms of $P(W,t)$ and $P(I,t)$ the probabilities respectively of feeling well and ill on day t. Show that

$$\frac{dP(W,t)}{dt} = 0.3 - 0.7P(W,t).$$

Solve this equation to find $P(W,t)$, given that initially $P(W,t) = 1.0$, and find the long-term value of $P(W,t)$.

[7 marks]

4. The number of bacteria, $n(t)$, in a certain culture satisfies the equation

$$\frac{dn(t)}{dt} = n^2 - 6n + 8$$

where the unit of n is 1,000 bacteria and the unit of t is 1 minute. Without solving this equation, show that if initially $n < 4$ then the number of bacteria stabilises at 2,000 . What happens if initially n is greater than 4?

[7 marks]

5. $\mathbf{r}(t)$ is the position vector of a particle at time t seconds. $\mathbf{a}(t)$ and $\mathbf{v}(t)$ are it's acceleration and velocity respectively. The former satisfies

$$\mathbf{a}(t) = 9\mathbf{i}\cos 3t + (18\mathbf{j} + 27\mathbf{k})\sin 3t.$$

Given that the particle was initially at rest at the point $2\mathbf{i}$, find both $\mathbf{v}(t)$ and $\mathbf{r}(t)$.

[6 marks]

6. A horse pulls a plough of mass m Kg in a straight line, exerting a force $(2+\sin 2t)$ Newtons at time t seconds. The resistance to the plough's motion is $[2-\exp(-2t)]$ Newtons. Write down Newton's equation of motion for the plough and use it to find both its velocity and the distance it has travelled by the time t seconds, given that it was initially at rest.

[8 marks]

7. A particle travels in the (x, y) plane, where the y-axis is vertical. It is thrown from the origin with velocity u m/s at an angle α to the horizontal x-axis. There is no air resistance. Show that if it is to go through the point (x, y) where $x > 0, y > 0$, then (x, y) has to satisfy the relation

$$y = x \tan \alpha - gx^2/(2u^2 \cos^2 \alpha).$$

[5 marks]

8. A particle of mass m Kg travels along the x-axis and is acted upon by a force $-mc/x^5$ Newtons where x is the distance in metres from the origin and $c > 0$, is a constant. Show that

$$v^2 = u^2 - \frac{c}{2a^4} + \frac{c}{2x^4},$$

where v is the velocity, u is the initial velocity (in metres/sec.) and $x = a$ is the initial position.

[6 marks]

Section B

9. The number of rats, $n(t)$, in a city at t months is increasing by 20% per month. It is decided to start, at $t = 0$, an extermination programme which will kill rats at a constant rate of b per month. Write down the differential equations satisfied by $n(t)$ for $n(t) > 0$, and for $n(t) = 0$.

[3 marks]

Without solving this equation show that , for the rat population to be completely eliminated, $b > n(0)/5$, where $n(0)$ is the initial rat population.

[4 marks]

Solve the equation, and show that when $b > n(0)/5$ the time required for the complete extermination of the rats is

$$5\ln(\frac{5b}{5b - n(0)})\text{months.}$$

[4 marks]

For $0 \leq t \leq 2$, sketch on the same axes $n(t)$ as a function of t for the cases: $b = n(0)/10$, and $b = 2n(0)$ for $0 \leq t \leq 2$, in particular showing the slopes and any zeros of the curves. (Hint: note the values of $\mathrm{d}n/\mathrm{d}t$.)

[4 marks]

10. A bowl contains jam whose initial volume is 300cc. Wasps are attracted to it at a rate $v(t)/100$ wasps per minute where $v(t)$ cc. is the volume of the jam at time t m. Each wasp eats 1/16 cc. of jam per minute.
Write down differential equations for $n(t)$, the number of wasps on the jam at time t and for $v(t)$.

[5 marks]

Initially there are no wasps on the jam. Derive a relation connecting $v(t)$ and $n(t)$. Sketch the complete graph of this relation. Indicate the realistic part of the graph and the direction in which the point $(n(t), v(t))$ changes with t.

[8 marks]

How many wasps are in the bowl when the jam is all eaten?

[2 marks]

11. Show that the acceleration $\mathrm{d}v/\mathrm{d}t$ of a particle can be written as $v\mathrm{d}v/\mathrm{d}s$ where v and s are the particle's velocity and position respectively.

[1 mark]

A particle of mass m Kg falls under gravity, its initial velocity being zero. Air resistance to the motion is mkv^2 Newtons where k is a constant. Show that its velocity after it has fallen s metres is

$$v = a\sqrt{1 - e^{-2ks}}$$

where $a = \sqrt{g/k}$.

[8 marks]

Show that

$$\frac{1}{a^2 - v^2} = \frac{1}{2a}(\frac{1}{a - v} + \frac{1}{a + v}).$$

[1 mark]

Show also that the velocity at time t sec. is

$$v = a(\frac{e^{2akt} - 1}{e^{2akt} + 1}).$$

[5 marks]

12.(i) A particle of mass m Kg is in equilibrium at the end of an elastic string. At time $t = 0$ it is disturbed so that its initial velocity is u m/s. The force due to the displacement, x, (i.e. the distance in metres of the particle from its equilibrium point) is $-m\omega^2 x$ Newtons, where ω^2 is a constant. Write down Newton's equation of motion and hence, or otherwise, write down the equation for the conservation of energy. (Note that the other forces cancel as the particle was in equilibrium.)

[5 marks]

Using Newton's equation of motion, or otherwise, calculate $x(t)$.

[4 marks]

(ii) Suppose now that there is an additional force $6mu\omega\cos(2\omega t)$ Newtons acting on the particle. Find the general solution for $x(t)$. If the initial conditions are as above, find the actual solution for $x(t)$.

[6 marks]