Candidates should attempt all questions in Section A and three questions in Section B.

Section A

1. (i) A 'hole in the wall' processes a person's demands in 1.25 minutes. The number of people arriving is constant, at one every minute. At 9.00a.m. there were no people in the queue. Writing $n(t)$, where n is taken to be real, as the number of people in the queue at the time t minutes after 9.00a.m., find a differential equation for $n(t)$ and solve it. How many people are in the queue at 9.40a.m.? [3 marks]

(ii) A queue at this 'hole in the wall' discourages people from joining it. So a better model for the rate at which people join the queue would be $(1-n(t)/50)$ people every minute. Write down the new differential equation for $n(t)$ and solve it, with the same initial condition. Sketch the graph of $n(t)$ against t.

[7 marks]

2. George invests \$ 600. per year in an account starting in year 0. Thus in year 1 and subsequent years he adds \$600. to this account. The interest on the account is 3% per year , which is also added to the account. The equation for the amount $s(n+1)$, in units of \$100., in the account in year $n+1$ in terms of $s(n)$ is

$$s(n+1) = 1.03s(n) + 6.$$

Explain how this equation was derived. Solve this equation to find $s(n)$ and calculate $s(5)$ to the nearest integer.

[5 marks]

3. A sixth form student wants to study mathematics at university, but cannot decide between Liverpool or Cambridge. If on one day she prefers Liverpool, she changes her mind at a rate so that the probability that next day she chooses Liverpool is 0.5 . If, however Cambridge is her choice on one day, the probability that next day she chooses Cambridge is 0.4 . Treat this problem as a two state process: writing $P(L,t), P(C,t)$ as the probabilities that at time t (in days) she chooses Liverpool and Cambridge respectively, write down the equations for $\frac{dP(L,t)}{dt}$ and $\frac{dP(C,t)}{dt}$, and derive the equation

$$\frac{dP(L,t)}{dt} = 1.1[\frac{6}{11} - P(L,t)].$$

Solve the equation for $P(L,t)$ given that at time $t = 0, P(C,0) = 0$. What is the long term value of $P(L,t)$?

[7 marks]

4. In a simplified model of the population, $n(t)$ (in units of 1,000,000), of fish in one of the North American lakes, $n(t)$ satisfies the equation

$$\frac{dn}{dt} = 2n(1 - \frac{n}{100}) - \frac{fn}{(5+n)}$$

where the last term represents the effect of fishing. When $f = 19$, show that $n = 5$ and $n = 90$ are equilibrium values of n. Are there any other equilibrium values and, if so, what are they? If initially $n = 4$, say what happens to $n(t)$ in the long-run; justify your answer.

[7 marks]

5. The position vector of a particle is written as $\mathbf{r}(\mathbf{t})$. Its velocity at time t is

$$6\mathbf{i} + 4\cos(2t)\mathbf{j} + 2\sin(2t)\mathbf{k}.$$

Its initial position is $\mathbf{k}$. Find $\mathbf{r}(t)$ and the acceleration of the particle.

[6 marks]

6. A porpoise, of mass m Kg., swims in a straight line. It exerts a driving thrust of $T(0.9 + 0.1\sin(2t))$ Newtons at time t, the oscillation being caused by the tail movement. The drag to the motion increases the further it swims, and can be written as $T(1 - e^{-t})$ Newtons. Write down Newton's equation of motion for the porpoise and use it to find its velocity, given that it starts from rest.

[7 marks]

7. Show that the acceleration of a particle travelling in a straight line can be written as $v\frac{dv}{ds}$, where v and s are the velocity and distance travelled.

A particle of mass m is thrown vertically upwards with initial velocity um/s. Writing the air resistance as mkv Newtons, show that the particle reaches a height $(u - (g/k)\ln(1 + ku/g))/k$.

[7 marks]

8. The angular momentum $\mathbf{h}$ of a particle about the origin is

$$m\mathbf{r} \times \dot{\mathbf{r}},$$

where m is the mass and $\mathbf{r}$ is the position vector of the particle.
(i) Writing $\mathbf{r}$ as $x\mathbf{i} + y\mathbf{j}$, express $\mathbf{h}$ in terms of $m, x, y, \dot{x}, \dot{y}, \mathbf{i}, \mathbf{j}, \mathbf{k}$.
(ii) $\mathbf{F}$ is the resultant force acting on the particle. Show that

$$\frac{d\mathbf{h}}{dt} = \mathbf{r} \times \mathbf{F}.$$

[6 marks]

Section B

9. Cars arrive from a particular direction at a junction at a constant rate of 8 per minute. The traffic lights are on red for 2.25 minutes. Show that the value of N, where N is the number of cars stopped by the traffic lights during a period when the lights are red is 18?

[1 mark]

(i) Suppose these cars leave the junction when the lights are green at the rate of 14 cars per minute. How many minutes do the lights have to be 'green' for 18 cars to be cleared? (*Assume that the lights only go 'red' and 'green'.)*

[2 marks]

(ii) Because, when the lights go 'green', the later cars are travelling faster than the earlier cars, the rate at which these cars leave the junction is more accurately written as $14(1 + \frac{n(t)}{70})$ where $n(t)$ is the number of cars in the queue at the lights at time t minutes after the lights go 'green'. Write down the differential equation satisfied by $n(t)$. Suppose initially there are N' cars in the queue. Find $n(t)$ in terms of N' and t.

[6 marks]

Suppose that the lights are 'green' for a period of $5\ln(5/4)$ minutes. Show that the traffic through the junction in this direction can be steady, i.e. the same number of cars leave during a period when the lights are 'green' as are stopped by the lights when they are 'red'. For this to happen, it may be that some cars are left at the lights when the lights change from 'green' to 'red'; if so, find the value of N'.

[6 marks]

10. Fred has a bacterial infection. The doctor gives him pills to try to kill off the bacteria. We write $v(t)$ for the number of bacteria and $m(t)$ for the amount of medicine (in suitable units) in Fred's body at time t. The effect of the medicine can be modelled by the equation, provided $v > 0$,

$$dv/dt = (v(t) - 3m).$$

The medicine is modelled by the equation,provided $v > 0$,

$$dm/dt = 32 - m(t) - v(t)$$

where the constant '32' represents the rate at which the pills are taken.

Solve these equations, either by following the instructions below or otherwise: (i)Take new variables V, M where $v = V + 24$ and $m = M + 8$ and rewrite the equations in terms of V and M.

[2 marks]

(ii) Derive a general solution to these new equations, by putting $V = Ae^{\lambda t}$ and $M = Be^{\lambda t}$.

[6 marks]

(iii) Given that the initial values are $v = 12$ and $m = 0$, sketch the graph of v against t and show that the infection continues to increase till $\frac{1}{4}\ln 3$ days after the first dose of medicine.

[7 marks]

11. A particle of mass m Kg. lies at the end of an elastic string with modulus $m\lambda$. The other end of the string is fixed to a point O and the string is extended by an amount x m in a horizontal direction, on a smooth horizontal plane.
(i) Write down Newton's equation of motion (neglecting any frictional effects) and from it derive the equation for conservation of energy

$$E = (m\dot{x}^2 + m\lambda x^2)/2.$$

[3 marks]

(ii) Taking frictional effects into account means that the rate of dissipation of energy due to them is $mk\dot{x}^2$. Use the above expression for E to get the equation

$$\ddot{x} + k\dot{x} + \lambda x = f.$$

where f is zero in this case.

[2 marks]

(iii) Let $k = 6$ and $\lambda = 25$. Solve the equation in (ii), with $f = 0$ and initial conditions $x = 2$ and $\dot{x} = 0$.

[5 marks]

(iv) Experimentally it was found that superimposed on the above solution was a term $x = (4\sin t - \cos t)/34$. Find the resulting form for f.

[5 marks]

12.(i) At a village fete, a farmer is asked to throw a wellie through a hoop which is 10m above his shoulder and a horizontal distance 20m away and then for the wellie to land 40m away at the same horizontal height as his shoulder. Taking the wellie to be a particle and the hoop as a point, show that the horizontal velocity u of the wellie has to be $(20g)^{\frac{1}{2}}$m/s and that it is thrown at an angle of $\frac{\pi}{4}$ to the horizontal.

[8 marks]

(ii)Now the hoop is moved to a point 10m above his shoulder at a horizontal distance 4m away. The farmer is asked to try to throw the wellie through the hoop. With the same assumptions and assuming that his throw has the same speed, at what angle(s) should he throw the wellie?
(Hint: $\sec^2\theta = 1 + \tan^2\theta$.)

[7 marks]