## SECTION A

**1.** Express the complex number

$$z = \frac{(4+i)(3+i)}{1+2i}$$

in the form $a + ib$.

Mark the points $z, \overline{z}$ and $z + \overline{z}$ on an Argand diagram.

[6 marks]

**2.** Find the modulus and argument of each of the complex numbers

$$z = -1 + 2i \text{ and } w = 4 + 3i.$$

Hence, or otherwise, find the modulus and argument of $z^2/w$.

[6 marks]

**3.** Find all solutions of the equation $(z-3)^3 = 8$, expressing each in the form $a + ib$. Mark the corresponding points on an Argand diagram.

[6 marks]

**4.** Use the formulae

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta}), \ \sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta}),$$

to verify the identities

$$2\cos\theta\sin\theta = \sin 2\theta,$$

and

$$4\cos^3\theta = \cos 3\theta + 3\cos\theta.$$

[6 marks]

**5.** Use simple iteration, with starting value $x_0 = 0.5$, to find a solution of the equation $x = e^{-x}$, correct to 3 decimal places.

[6 marks]

**6.** Find the solution of the differential equation

$$2x^2 \frac{dy}{dx} = e^y,$$

which satisfies $y(0) = 1$.

[6 marks]

**7.** Find the general solution of the differential equation

$$\frac{dy}{dx} + 3\frac{y}{x} = x^{-\frac{1}{2}}.$$

[6 marks]

**8.** Find the general solution of the differential equation

$$2\frac{d^2y}{dx^2} + 3\frac{dy}{dx} - 2y = 7e^{-3x}.$$

[6 marks]

**9.** Find all first order partial derivatives of the functions

$$f(x,y) = x^2y - xy^2, \quad g(x,y) = x\ln(xy),$$

$$h(x,y,z) = 2\cos(x^2 + yz).$$

[7 marks]

## SECTION B

**10.** A function $f(x)$, satisfies

$$f(0) = 1, \;\; f'(0) = -\frac{1}{2}, \;\; f''(0) = \frac{1}{3}, \;\; f'''(0) = -\frac{1}{4}.$$

Use these values in a truncated Maclaurin series to find a quadratic approximation, $F(x)$,to $f(x)$, valid near $x = 0$. Similarly, write down a cubic approximation, $C(x)$.
Calculate $\alpha = F(0.5)$ and $\beta = C(0.5)$.
A second function is defined by $g(x) = \ln(f(x))$. Show that

$$g'(x) = \frac{f'(x)}{f(x)}, \;\; g''(x) = \frac{f(x)f''(x) - (f'(x))^2}{f(x)^2}.$$

Use these results to find a quadratic approximation, $G(x)$, to $g(x)$, valid near to $x = 0$.
Use a calculator to compute $G(0.5)$, $\ln\alpha$ and $\ln\beta$.
Comment on the discrepancies between these values.

[15 marks]

**11.** Find the solution of the differential equation

$$\frac{d^2y}{dx^2} - 3\frac{dy}{dx} + 2y = F(x),$$

where $y(0) = 0$ and $y'(0) = 1$, in each of the cases

$$F(x) = 0; \;\; F(x) = \cos(x) + 3\sin(x).$$

[15 marks]

**12.** (i) In a coal processing plant, the flow rate $V$ of slurry along a pipe of radius $r$ and length $l$ is given by

$$V = \frac{\pi p r^4}{8 \eta l},$$

where $p$ is the pressure difference and $\eta$ measures viscosity.
If $r$ and $l$ are known to within 2%, $p$ is known to within 5%, but $\eta$ is only known to within 10%, find the percentage by which the computed value of $V$ may be in error.

(ii) Show that the wave equation

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2},$$

where $c$ is a constant, is satisfied by

$$u(x,t) = (x + ct)\sin(x + ct) - (x - ct)^2.$$

[15 marks]

**13.** The function $\cos(x^2)$ can be approximated by the Maclaurin polynomial

$$M(x) = 1 - \frac{x^4}{2} + \frac{x^8}{24},$$

for small $x$, measured in radians.
By how much does $M(x)$ differ from $\cos(x^2)$ at $x = 0, 0.25, 0.5, 0.75$ and $1$?
Use the Trapezium Rule and Simpson's Rule, each with five ordinates (four strips), to find approximate values, $T$ and $S$, respectively, of

$$I = \int_0^1 \cos(x^2)dx.$$

Evaluate

$$M = \int_0^1 M(x)dx,$$

by explicit integration. Compare $T, S$ and $M$ with the accurate value $I = 0.904524\cdots$.

[15 marks]

**14.** Show that the isoclines for the differential equation

$$\frac{dy}{dx} - y^2 = x^2 - 3,$$

are circles, centred on the origin.
Sketch the $k = 6, 1, 0, -2$ and $-3$ isoclines, and hence indicate the slope field.
Sketch the solution, $y_0(x)$, of the differential equation which passes through the origin.
Show, by a Taylor series method, that a cubic approximation to $y_0(x)$ is given by

$$Y_0(x) = -3x + \frac{10x^3}{3}.$$

With reference to your sketches, explain why $Y_0(x)$ cannot be a good approximation to $y_0(x)$ for all $x$.

[15 marks]