2MA1C Maths for Cilvil Engineers September 1999

Candidates should attempt the whole of Section A and THREE questions from Section B. Section A carries 52% of the available marks.

Note: **i**, **j** and **k** denote unit vectors along the positive x, y and z axes respectively.

SECTION A

1. Solve the quadratic equation $x^2 - 8x + 13 = 0$. Hence find the values of x which satisfy the equation

$$\frac{3}{x-1} + \frac{1}{x-5} = 2.$$

[4 marks]

2. The function f is defined by

$$f(x) = \frac{2x+5}{x+3} \qquad x \neq -3.$$

Find the inverse function $f^{-1}(x)$ and verify that $f[f^{-1}(x)] = x$. [4 marks]

3. The vectors $\mathbf{a}$ and $\mathbf{b}$ are the position vectors (relative to the origin O) of the points $A(1,2,1)$ and $B(0,1,1)$. What is the angle between $\mathbf{a}$ and $\mathbf{b}$?

[4 marks]

4. What is the equation of the plane with normal $\mathbf{i} - 4\mathbf{j} + \mathbf{k}$ which passes through the point $(3,1,4)$? What is its perpendicular distance from the origin?
[7 marks]

5. Simplify

$$\ln 45 - \ln 20 + 2\ln\left(\frac{2}{3}\right).$$

[4 marks]

6. Sketch the graphs in the xy plane represented by the following equations in polar co-ordinates:

$$\text{(i)} \quad r = \frac{1}{4}\theta + 1, \quad 0 \leq \theta \leq 2\pi, \qquad \text{(ii)} \quad r = 2, \quad -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}.$$

[4 marks]

7. State L'Hôpital's rule for the evaluation of limits. Hence or otherwise evaluate

$$\lim_{x \to 0} \frac{x^2 - \cos 2x + 1}{\cosh 2x - \cosh x}.$$

[6 marks]

8. Differentiate the following functions with respect to x:

(i) $x^4 \sin x$, (ii) $\cos^5 x$, (iii) $\dfrac{1+\sinh x}{1+\cosh x}$, (iv) e^{-x^4}.

[5 marks]

9. Given that $x^3 \sin y + \cos x = e^y$, find $\frac{dy}{dx}$ as a function of x and y.

[3 marks]

10. Evaluate the following integrals:

(i) $\displaystyle\int_0^{\frac{\pi}{2}} x \cos x \, dx$, (ii) $\displaystyle\int_0^1 \frac{x^3}{\sqrt{x^4+1}} \, dx$, (iii) $\displaystyle\int_3^5 \frac{3x+4}{(x-2)(x+3)} dx$.

[11 marks]

SECTION B

11. Find and classify all stationary points of the function f defined by

$$f(x) = x - 7 + \frac{4}{x-2}, \qquad x \neq 2.$$

Sketch the graph of $y = f(x)$, showing clearly the turning points, asymptotes and the points at which the graph intersects the x and y axes. What is the equation of the tangent to the curve at $x = 1$? [16 marks]

12. Find the equation of the plane which passes through the points A, B and C with co-ordinates $A(3,0,3)$, $B(4,4,4)$ and $C(3,2,4)$. Show that the perpendicular distance of this plane from the origin is 4 units.

What is the equation of the line perpendicular to the plane and passing through the point A? Does this line intersect the line through B in the direction $\mathbf{i} - 3\mathbf{j} + \mathbf{k}$? [16 marks]

13. Write down the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x}. Show that

$$2\sinh x \cosh x = \sinh 2x, \qquad 1 + 2\sinh^2 x = \cosh 2x.$$

Hence show that

$$\sinh 4x = 4(\sinh x + 2\sinh^3 x)\cosh x.$$

The inverse hyperbolic sine function $y = \sinh^{-1} x$ is defined by the equation $\sinh y = x$. Show that $e^{2y} - 2xe^y - 1 = 0$, and hence prove that

$$\sinh^{-1} x = \ln(x + \sqrt{x^2+1}).$$

Hence or otherwise show that

$$\frac{d}{dx}(\sinh^{-1} x) = \frac{1}{\sqrt{x^2+1}}.$$

[16 marks]

14. The framework supporting a bridge contains three thin linear struts AB, CD and EF, where the cartesian co-ordinates of the points A, B, C, D, E and F are given by $A(2,3,0)$, $B(8,4,6)$, $C(4,6,1)$, $D(8,7,4)$, $E(1,1,0)$ and $F(3,2,2)$. Find vector equations for the straight lines AB, CD and EF. Hence calculate the perpendicular distance between AB and CD. Also calculate the perpendicular distance of EF from the origin.

[You may assume that if the vector equations for two non-parallel straight lines L_1 and L_2 are respectively:

$$\mathbf{r} = \mathbf{a} + \lambda\mathbf{u}, \qquad \text{and} \qquad \mathbf{r}' = \mathbf{c} + \mu\mathbf{v},$$

where λ and μ are variable scalar parameters, then the perpendicular distance d between L_1 and L_2 is given by

$$d = \frac{|(\mathbf{a} - \mathbf{c}).(\mathbf{u} \times \mathbf{v})|}{|\mathbf{u} \times \mathbf{v}|}.]$$

[16 marks]

15. The curve C is the section of the graph $y = x^3$ which lies between $x = 0$ and $x = 1$. Draw carefully a sketch showing C and the straight line $y = x$ on the same diagram. Show that the area enclosed between the curve C and the straight line has the value $\frac{1}{4}$ units.

Show that the volume of the solid generated by rotating the area between C and the x axis through 360° about the x axis is $\frac{1}{7}\pi$ units.

Finally, show that the area S of the curved surface of this solid is given by

$$S = 2\pi \int_0^1 x^3 \sqrt{1 + 9x^4}\, dx,$$

and hence evaluate S. [16 marks]