2MA1C June 1998

Candidates should attempt the whole of Section A and THREE questions from Section B. Section A carries 52% of the available marks.

Note: **i**, **j** and **k** denote unit vectors along the positive x, y and z axes respectively.

SECTION A

1. Solve the quadratic equation $x^2 - 2x - 8 = 0$. Hence find the values of x which satisfy the equation $e^{2x} - 2e^x - 8 = 0$. [5 marks]

2. The function f is defined by $f(x) = \frac{2x-1}{x+4}$, $x \neq -4$. Find the inverse function $f^{-1}(x)$ and verify that $f[f^{-1}(x)] = x$. [4 marks]

3. Sketch the graphs in the xy plane represented by the following equations in polar co-ordinates:

$$\text{(i)} \quad r = \frac{1}{2}\theta + 1, \quad 0 \leq \theta < 3\pi, \qquad \text{(ii)} \quad r = 2\sin\theta, \quad 0 \leq \theta < \pi.$$

[5 marks]

4. The three vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ are the position vectors (relative to the origin O) of the points A, B and C with co-ordinates $A(2, -2, 1)$, $B(4, 0, -3)$ and $C(1, 2, -3)$. Calculate

$$\text{(i)} \quad \mathbf{a}.\mathbf{b}, \qquad \text{(ii)} \quad \mathbf{b} \times \mathbf{c}, \qquad \text{(iii)} \quad \mathbf{a}.(\mathbf{b} \times \mathbf{c}).$$

Hence find the angle between $\mathbf{a}$ and $\mathbf{b}$, the area of the parallelogram with edges parallel to $\mathbf{b}$ and $\mathbf{c}$ and the volume of the parallelepiped with edges parallel to $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$. [7 marks]

5. What is the equation of the plane whose normal is $\mathbf{i} - 2\mathbf{j} + \mathbf{k}$ and which passes through the point $(1, -4, -3)$? What is the perpendicular distance of this plane from the origin? [6 marks]

6. State L'Hôpital's rule for the evaluation of limits. Hence or otherwise evaluate

$$\lim_{x \to 0} \frac{1 - \cos x}{\cos x - \cos 2x}.$$

[6 marks]

7. Differentiate the following functions with respect to x:

$$\text{(i)} \quad \cos(x^5), \qquad \text{(ii)} \quad e^{\sin x}, \qquad \text{(iii)} \quad \frac{x^2}{1 - x^3}, \qquad \text{(iv)} \quad \sinh^3 x.$$

[6 marks]

8. Given that $y = x^3 \sin y$, find $\frac{dy}{dx}$ as a function of x and y. [3 marks]

9. Evaluate the following integrals:

(i) $\displaystyle\int_0^{\frac{\pi}{2}} x \sin x \, dx,$ (ii) $\displaystyle\int_0^{\frac{\pi}{4}} \frac{\cos 2x}{2 + \sin 2x} dx,$ (iii) $\displaystyle\int_4^{6} \frac{x + 7}{(x + 2)(x - 3)} dx.$

[10 marks]

SECTION B

10. Find and classify all stationary points of the function f defined by

$$f(x) = x - 1 + \frac{1}{x-3}, \qquad x \neq 3.$$

Sketch the graph of $y = f(x)$, showing clearly the turning points, asymptotes and the points at which the graph intersects the x and y axes. Show that the tangent to the graph at $x = 6$ passes through the origin. [16 marks]

11. Find the equation of the plane which passes through the points A, B and C with co-ordinates $A(6,1,4)$, $B(4,-1,3)$ and $C(2,-4,1)$. Show that the perpendicular distance of this plane from the origin is 4 units.

Find the perpendicular distance of the point $P(3,1,1)$ from the plane. A straight line is drawn from P perpendicular to the plane, meeting it at Q. Write down the vector from P to Q, and hence find the co-ordinates of Q. [16 marks]

12. Write down the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x}. Hence prove that $2\cosh^2 x - 1 = \cosh 2x$, and show that

$$\cosh 4x = 8\cosh^4 x - 8\cosh^2 x + 1.$$

The inverse hyperbolic cosine function $y = \cosh^{-1} x$ is defined by the equation $\cosh y = x$, where $y \geq 0$ and $x \geq 1$. Show that $e^{2y} - 2xe^y + 1 = 0$, and hence prove that

$$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1}).$$

Hence or otherwise show that

$$\frac{d}{dx}(\cosh^{-1} x) = \frac{1}{\sqrt{x^2 - 1}}.$$

[16 marks]

13. The vector equations for two non-parallel straight lines L_1 and L_2 are respectively:

$$\mathbf{r} = \mathbf{a} + \lambda \mathbf{u}, \qquad \text{and} \qquad \mathbf{r}' = \mathbf{c} + \mu \mathbf{v},$$

where λ and μ are variable scalar parameters.

Show that the perpendicular distance d between L_1 and L_2 is given by

$$d = \frac{|(\mathbf{a} - \mathbf{c}).(\mathbf{u} \times \mathbf{v})|}{|\mathbf{u} \times \mathbf{v}|}.$$

The framework supporting a bridge contains two thin linear struts AB and CD, where the cartesian co-ordinates of the points A, B, C and D are given by $A(1, 3, 4)$, $B(7, 9, 5)$, $C(2, 5, 7)$ and $D(5, 9, 8)$. Find vector equations for the straight lines AB and CD. Hence calculate the perpendicular distance between AB and CD. [16 marks]

14. The curve C is the section of the graph $y = 2x^{\frac{1}{2}}$ which lies between $x = 0$ and $x = 1$. Draw carefully a sketch showing C and the straight line $y = 2x$ on the same diagram. Prove that the area enclosed between C and the straight line has value $\frac{1}{3}$ unit.

Show that the volume of the solid generated by rotating the area between C and the x-axis through 360° about the x axis is 2π units.

Finally, show that the surface area S of the curved surface of this solid is given by

$$S = 4\pi \int_0^1 \sqrt{x+1}\, dx,$$

and hence evaluate S. [16 marks]