2MA1C MATHEMATICS I FOR CIVIL ENGINEERS JANUARY 1999

Candidates should attempt the whole of Section A and THREE questions from Section B. Section A carries 52% of the available marks.

Note: $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$ denote unit vectors along the positive x, y and z axes respectively.

SECTION A

1. Solve the quadratic equation $x^2 - 4x + 2 = 0$. Hence find the values of x which satisfy the equation

$$\frac{3}{x-3} - \frac{1}{x-1} = 2x.$$

[5 marks]

2. The function f is defined by

$$f(x) = \frac{4x+1}{x-3} \qquad x \neq 3.$$

Find the inverse function $f^{-1}(x)$ and verify that $f^{-1}[f(x)] = x$. [4 marks]

3. The vectors **a**, and **b** are the position vectors (relative to the origin O) of the points A and B respectively, with co-ordinates $A(1, 1, -2)$ and $B(-1, 2, -1)$. Calculate

$$\text{(i)} \quad \mathbf{a}.\mathbf{b}, \qquad \text{(ii)} \quad \mathbf{a} \times \mathbf{b}.$$

[3 marks]

4. Write down the vector equation of the straight line through the points $(2, -3, 2)$ and $(4, -7, 6)$. Compute the perpendicular distance from the origin to this straight line. [7 marks]

5. By writing $\frac{7\pi}{12} = \frac{\pi}{4} + \frac{\pi}{3}$, and without using a calculator, compute $\sin\left(\frac{7\pi}{12}\right)$. (Show your working.) [3 marks]

6. Sketch the graphs in the xy plane represented by the following equations in polar co-ordinates:

$$\text{(i)} \quad r\cos\theta = 1, \quad -\frac{\pi}{4} \leq \theta \leq \frac{\pi}{4}, \qquad \text{(ii)} \quad r = 2(\cos\theta + 1), \quad 0 \leq \theta < 2\pi.$$

[5 marks]

7. State L'Hôpital's rule for the evaluation of limits. Hence or otherwise evaluate

$$\lim_{x \to 0} \frac{x^2}{\cosh 3x - \cos x}.$$

[6 marks]

8. Differentiate the following functions with respect to x:

(i) $e^x \cos x$, (ii) $\sin^3 x$, (iii) $\dfrac{4 - x^2}{x^2 - 5}$, (iv) $\tan(x^3)$.

[5 marks]

9. Given that $x^2y^3 = \cos y$, find $\frac{dy}{dx}$ as a function of x and y. [3 marks]

10. Evaluate the following integrals:

(i) $\displaystyle\int_1^2 x^3 \ln x \, dx$, (ii) $\displaystyle\int_0^1 \frac{1}{\sqrt{x}} e^{\sqrt{x}} \, dx$, (iii) $\displaystyle\int_3^5 \frac{x - 5}{(x + 1)(x - 2)} dx$.

[11 marks]

SECTION B

11. Find and classify all stationary points of the function f defined by

$$f(x) = x - 6 + \frac{4}{x-1}, \qquad x \neq 1.$$

Sketch the graph of $y = f(x)$, showing clearly the turning points, asymptotes and the points at which the graph intersects the x and y axes. Find where the curve intersects the straight line $y = x - 7$. [16 marks]

12. Find the equation of the plane which passes through the points A, B and C with co-ordinates $A(3, -3, 3)$, $B(5, -1, 2)$ and $C(4, -5, 1)$. Show that the perpendicular distance of this plane from the origin is 5 units.

Check that the plane $x - 2y + 3z = 18$ also passes through the point A. What is the equation of the straight line formed by the intersection of the two planes? [16 marks]

13. Write down the definitions of $\sinh x$ and $\cosh x$ in terms of e^x and e^{-x}. Show that

$$2 \sinh x \cosh x = \sinh 2x, \qquad \cosh^2 x + \sinh^2 x = \cosh 2x.$$

Hence derive an expression for $\tanh 2x$ in terms of $\tanh x$.

The inverse hyperbolic tangent function $y = \tanh^{-1} x$ is defined by the equation $\tanh y = x$, where $-1 < x < 1$. Prove that

$$\tanh^{-1} x = \frac{1}{2} \ln\left(\frac{1+x}{1-x}\right).$$

Hence or otherwise show that

$$\frac{d}{dx}(\tanh^{-1} x) = \frac{1}{1-x^2}.$$

[16 marks]

14. A force $\mathbf{F}$ is applied to a rigid body at a point with position vector $\mathbf{r}$ relative to the origin O. Show that the magnitude of the vector $\mathbf{M} = \mathbf{r} \times \mathbf{F}$ gives the magnitude of the turning moment of $\mathbf{F}$ about O and the direction of $\mathbf{M}$ gives the axis of the rotation which $\mathbf{F}$ would produce.

A rigid body contains the points P_1, P_2, and P_3 with cartesian co-ordinates $P_1(1, 2, -1)$, $P_2(2, 1, -1)$ and $P_3(1, 0, -1)$. Write down the position vectors $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ of P_1, P_2, and P_3 respectively. A force $\mathbf{F}_1 = \mathbf{i} + 4\mathbf{j} - \mathbf{k}$ is applied at P_1 and a force $\mathbf{F}_2 = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$ is applied at P_2. Show that their total turning moment about O is given by

$$(\mathbf{r}_1 \times \mathbf{F}_1) + (\mathbf{r}_2 \times \mathbf{F}_2) = \mathbf{i} + \mathbf{j} + \mathbf{k}.$$

Additional forces $\mathbf{F}_3 = \lambda(\mathbf{i} - 2\mathbf{j} + \mathbf{k})$, applied at P_3, and $\mathbf{F}_4$, applied at O, also act on the body. Assuming that the body is maintained in static equilibrium under the action of all four forces, find the value of λ and show that

$$\mathbf{F}_4 = -\frac{1}{2}(9\mathbf{i} + 8\mathbf{j} - 5\mathbf{k}).$$

[16 marks]

15. Show that

$$1 + \tan^2\theta = \sec^2\theta.$$

The curve C is the section of the graph $y = \ln\cos x$ which lies between $x = 0$ and $x = \frac{\pi}{3}$. Show that the length L of C is given by

$$L = \int_0^{\frac{\pi}{3}} \sec x \, dx.$$

Show that if we write $t = \tan\frac{1}{2}x$, then we have

$$\cos x = \frac{1-t^2}{1+t^2}$$
$$\frac{dx}{dt} = \frac{2}{1+t^2}.$$

Hence show that

$$L = \ln\left(\frac{\sqrt{3}+1}{\sqrt{3}-1}\right).$$

[16 marks]