Full marks can be obtained for correct answers to **<u>FOUR</u>** questions.

Data provided: New Cambridge Elementary Statistical Tables by D.V. Lindley and W.F. Scott

Some useful Formulae

1) For any two events A and B

$P(A\cup B) = P(A) + P(B) - P(A\cap B)$,

$P(A\cap B) = P(A \mid B)\, P(B)$,

$P(A\cap \bar{B}) = P(A) - P(A\cap B)$.

2) For three events A, B and C

$P\{A\cap(B\cup C)\} = P\{(A\cap B) \cup (A\cap C)\}$,

$P\{A\cap(\bar{B}\cup \bar{C})\} = P(A) - P\{A\cap(B\cup C)\}$.

3) For two event T and D with complements $\bar{T}$ and $\bar{D}$, respectively,

$$P(D|\,T) \;=\; \frac{P(T|\,D)\,P(D)}{P(T|\,D)\,P(D) \;+\; P(T|\,\overline{D})\,P(\overline{D})}.$$

4) If X has a Binomial distribution with parameters n and p

$$P(X = x) \;=\; \binom{n}{x} p^{x}\,(1-p)^{n-x} \qquad (x \;=\; 0,1,\ldots,n)$$

and $E(X) = np$, $V(X) = np(1-p)$.

5) If X has a Poisson distribution with mean λ

$$P(X = x) \;=\; \frac{\lambda^{x}}{x!}\;\exp(-\lambda) \qquad (x \;=\; 0,\,1,\,\ldots),$$

and

$E(X) = \lambda$, $V(X) = \lambda$.

6) If X_1 and X_2 are independent Normal variables with means μ_1 and μ_2 and variances σ_1^2 and σ_2^2, respectively, then $Y = X_1 + X_2$ is also a Normal random variable with mean $\mu_1 + \mu_2$ and variance $\sigma_1^2 + \sigma_2^2$.

1. A family with two pre-school children, called Sarah and Jane, will watch a particular afternoon TV programme only if the mother (M) does not object to watching the programme and either or both Sarah (S) and Jane (J) do not object as well. Let M denote the event that the mother does not object to watching the programme and define S and J in the same way. It is know that

$P(S) = 0.23$, $P(J) = 0.25$, $P(M) = 0.5$,

$P(S \cap J) = 0.06$, $P(S \cap M) = 0.125$, $P(J \cap M) = 0.12$,

$P(S \cap J \cap M) = 0.029$.

Find the probability that

a) Sarah or Jane or both do not object to watching the programme. [4 marks]

b) Jane does not object to watching the programme but Sarah does. [5 marks]

c) The family watches the TV programme. [7 marks]

d) The mother does not object to watching the programme but both Sarah and Jane do. [9 marks]

2. A diagnostic test for a certain disease has two possible results, positive or negative. The test has a sensitivity of 99% (that is, for a person with the disease the test is positive with probability 0.99). For a person not having the disease the test is negative with probability 0.98 (the specificity). One percent of the population has the disease.

For one person chosen at random, the test is positive. Show that the probability the person has the disease is one in three. [17 marks]

If the test is positive, a patient is invited to retake the test; invariably the invitation is accepted. Assuming the above sensitivity and specificity still hold (but that now a priori probability of disease is 1/3), show that, if the second test is also positive, the probability that the person has the disease is now more than 95%. [8 marks]

3. In a series of 7 independent trials, the probability of a `success' is constant from trial to trial and equals p. Write down an expression for the probability that

i) exactly one success is observed,

ii) no more than one success is observed. [5 marks]

A scientist inoculates seven mice, one at a time, with a disease germ and after an appropriate time has elapsed, she examines each mouse individually and records whether or not it has contracted the disease. Suppose that the probability of contracting the disease from the inoculated germ is $^1/_6$ for each mouse and that the mice contract the disease independently of each other. What is the probability that

iii) no more than one mouse has contracted the disease,

iv) at least four mice have contracted the disease. [8 marks]

A scientist at a different laboratory continues to inoculate the mice with the disease germ until she finds two mice that have developed the disease. It may be assumed that at this laboratory also the probability that an inoculated mouse contracts the disease is $^1/_6$ for each mouse and that the mice contract the disease independently of each other. Find the probability that

v) exactly two mice are required,

vi) exactly three mice are required, [12 marks]

4. A news vendor orders four copies of a certain magazine per week. The number of individuals who come in to buy this magazine in a week has a Poisson distribution with mean 4. Find the probability distribution of the random variable, Y, where Y is the number of copies of the magazine that are actually sold in a week. [17 marks]

Deduce that the expected value, E(Y), of Y equals 3.2. [8 marks]

5. The time (in seconds) taken by a motor to start after it has been switched on is a continuous random variable with probability density function

$$f(x) = \begin{cases} \left(\frac{3}{2}\right)\left(\frac{1}{x^2}\right) & 1 \le x \le 3, \\ 0 & elsewhere. \end{cases}$$

Verify that f(x) defines a valid probability density function. [8 marks]

Find the probability that the motor starting time is

a) less than 1.5 seconds,

b) greater than 2.5 seconds. [10 marks]

If the motor takes more than 1.5 seconds to start, a light comes on and stays on until either one second has elapsed or until the motor starts (whichever happens first). Let the random variable Y denote the length of time the light stays on.

Find

i) P(Y = 0),

ii) P(Y = 1). [7 marks]

6. The employees of a building management firm are being offered a profit-sharing/salary package.

Suppose that an individual currently received an annual salary of £B. Under the scheme, this salary would be replaced by a basic salary of £(0.8)B plus a contribution £X_1 related to the firm's profitability over the next twelve months. Current forecasts predict that £X_1 will be £(0.4B). The uncertainties of future operations imply that £X_1 should be regarded as a random quantity, normally distributed, with expected value £(0.4)B and standard deviation £(0.2)B.

Evaluate the probability that over a single year an employee accepting the new scheme would have total earnings less than the present earnings. [12 marks]

The profitability in a subsequent year may be taken as contributing a further £X_2 under the scheme, £X_2 having the same distribution as £X_1; it has also been proposed that, in the second year, the basic salary should be increased to £(0.85)B. Under the present arrangements, on the other hand, an employee's salary in the second year is expected to go up to £(1.1)B. On the assumption that X_2 is distributed independently of X_1, evaluate the probability that over a two-year period, an employee accepting the new scheme would be worse off than under present arrangements. [13 marks]