mm31 jan99

Instructions to candidates

Answer FOUR questions.

In this paper **i**, **j** and **k** represent unit vectors parallel to the x, y and z axes respectively.

1. (a) If $\phi(x, y, z) = 3x^2y - y^3z^2$, find $\nabla\phi$ at the point $(1, -2, -1)$.

[5 marks]

(b) Given

$$\mathbf{V} = (x + 5y^2)\mathbf{i} + (y - 4z)\mathbf{j} + (x^3 + az)\mathbf{k},$$

determine the constant a so that $\nabla.\mathbf{V} = 0$.

[3 marks]

(c) Given

$$\mathbf{A} = x^2y\mathbf{i} - 2xz^2\mathbf{j} + 2xyz\mathbf{k},$$

find $\nabla \times (\nabla \times \mathbf{A}), \nabla(\nabla.\mathbf{A})$ and $\nabla^2\mathbf{A}$, where, in Cartesian coordinates,

$$\nabla^2\mathbf{A} = (\nabla^2 A_x)\mathbf{i} + (\nabla^2 A_y)\mathbf{j} + (\nabla^2 A_z)\mathbf{k}.$$

Verify that

$$\nabla \times (\nabla \times \mathbf{A}) = \nabla(\nabla.\mathbf{A}) - \nabla^2\mathbf{A}.$$

[17 marks]

2. Stokes's theorem for vector integrals states that for any vector field $\mathbf{A}$,

$$\oint_C \mathbf{A}.\mathbf{dr} = \int_S \mathrm{curl}\mathbf{A}.\mathbf{n}\, dS,$$

where the simple closed curve C is the boundary of the open surface S, and $\mathbf{n}$ is the unit normal to S. Indicate how the direction of the normal is related to the direction of traversal of C.

Calculate curl$\mathbf{A}$ for a vector of the form

$$\mathbf{A} = M(x,y)\mathbf{i} + N(x,y)\mathbf{j}.$$

Using this result and Stokes's theorem, or otherwise, prove Green's theorem, which states that

$$\oint_C (M\,dx + N\,dy) = \int\int_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

where S is a closed region of the xy plane bounded by C.

[7 marks]

Verify Green's theorem for the case $M = x^2 - y^2$, $N = x^2 + y^2$, when the region S is the triangle bounded by the lines $x = 1$, $y = 0$ and $x = y$.

[18 marks]

3. The divergence theorem for vector integrals states that for any vector field $\mathbf{A}$,

$$\int_S \mathbf{A}.\mathbf{n}\, dS = \int_V \operatorname{div}\mathbf{A}\, dV,$$

where the volume V is enclosed by the surface S.

(a) Given $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$, calculate $\operatorname{div}\mathbf{r}$. Verify the divergence theorem for the case $\mathbf{A} = \mathbf{r}$, for the cases

(i) A sphere of radius a, centre the origin.

(ii) A cube with sides of length a and one corner at the origin, choosing axes along the three edges meeting at O.

[22 marks]

(b) Calculate $\operatorname{div}\mathbf{E}$, where $\mathbf{E} = \mathbf{r}/r^3$. Comment on the result, with reference to Gauss's theorem for the electric flux through a closed surface.

[3 marks]

(For a vector field of the form $\mathbf{A} = f(r)\mathbf{r}$, if $r \neq 0$ then

$$\operatorname{div}\mathbf{A} = \frac{1}{r^2}\frac{d}{dr}\left[r^3 f(r)\right]).$$

4. The temperature $T(x,t)$ of a uniform rod lying on the positive x-axis from $x=0$ to $x=L$ satisfies the equation:

$$\frac{\partial T}{\partial t} = \alpha^2 \frac{\partial^2 T}{\partial x^2},$$

where α is a constant. The ends of the rod are kept at $T=0$ for all t, and at $t=0$ the initial temperature distribution is given by $T(x,0)=\phi(x)$. Given also that $T(x,t)\to 0$ as $t\to\infty$, use the Method of Separation of Variables to show that a solution which satisfies the boundary conditions is

$$T(x,t) = \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2\alpha^2}{L^2}t} \sin\left(\frac{n\pi x}{L}\right)$$

where

$$A_n = \frac{2}{L}\int_0^L \phi(x)\sin\left(\frac{n\pi x}{L}\right)\, dx.$$

[16 marks]

Given that

$$\phi(x) = T_0 \quad \text{for} \quad 0 < x < \frac{L}{2}$$

$$\phi(x) = 0 \quad \text{for} \quad \frac{L}{2} < x < L$$

where T_0 is a constant, show that

$$A_n = \frac{2T_0}{n\pi}\left[1 - \cos\left(\frac{n\pi}{2}\right)\right]$$

and evaluate A_1, A_2, A_3 and A_4.

[9 marks]

5. In plane polar coordinates, Laplace's equation in two dimensions takes the form

$$\frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r}\frac{\partial \Phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Phi}{\partial \theta^2} = 0. \tag{1}$$

Using the method of separation of variables, show that all single-valued solutions of this equation in the region $0 < a < r < b$ (where a and b are constants) which are of the form

$$\Phi = F(r)G(\theta)$$

are given by

$$\Phi = A + B\ln r + \sum_{n=1}^{\infty}\left(A_n r^n + \frac{B_n}{r^n}\right)(C_n \cos n\theta + D_n \sin n\theta)$$

where A, B, A_n, B_n, C_n, D_n are constants.

[18 marks]

The potential function Φ satisfies Eq. (1) in the region $0 < a < r < b$ and takes the values $\Phi = 0$ on $r = a$ and $\Phi = p\cos\theta$ on $r = b$ (where p is a constant). Show that

$$\Phi = \frac{pb}{b^2 - a^2}\left(r - \frac{a^2}{r}\right)\cos\theta.$$

[7 marks]