instruction to candidates

(time allowed 2 hours)

- candidates will be assessed on their best four answers
- if you attempt to answer more than the required number of questions, the marks awarded for the excess questions will be discarded (starting with your lowest mark)

Question 1

1.A What are the values of:

$$11^0, 11^1, 11^2, 11^3, 11^4 \quad ?$$

Why are these numbers easy to compute for a person who knows binomial coefficients? And why then does computing 11^5 become more tricky? **[15 marks]**

1.B For which values of $0 \leq k \leq n$ is

$$\binom{n}{k-1} \leq \binom{n}{k} \quad ?$$

Which value among $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}, \binom{n}{n}$ is the largest? **[10 marks]**

Question 2

2.A Prove, using mathematical induction, that $T(n) = \frac{3^n - 1}{2}$, where value $T(n)$ is defined by the recurrence:

$$T(n) = \begin{cases} 1 & \text{for } \; n = 1, \\ 3T(n-1) + 1 & \text{for } \; n > 1, \end{cases}$$

[15 marks]

2.B What is the value of:

$$\sum_{k=1}^{n} (4k + 2) \quad ?$$

Justify your answer. **[10 marks]**

Question 3

3.A Design and write pseudocode finding a *minimum* and a *maximum* element in array $A[1,\dots,n]$ of real numbers. The pseudocode should report both extremal values. What is the exact complexity of your solution? **[15 marks]**

3.B Which function is larger for almost all n (i.e. which function has higher order):

(i)

$$n^{\frac{1999}{2000}} \qquad \frac{n}{\log(n)}$$

(ii)

$$\log(n^{2000}) \qquad n$$

(iii)

$$n^{\log(n)} \qquad 2^{2000\log(n)}$$

[6 marks]

3.C What is the value of:

$$\log^*(256)$$

[2 marks]

3.D Which exact complexity is better for realistic values of parameter n,

$$2000n \quad \textit{or} \quad n\log(n)?$$

Justify briefly your answer. **[2 marks]**

Question 4

4.A Prove that

$$5n^2 + 2 = \Theta(n^2).$$

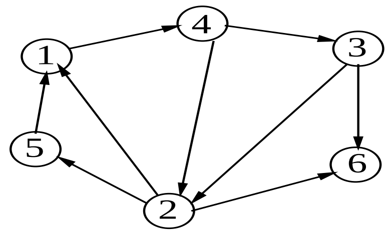
[15 marks]

4.B List all permutations of the sequence A B C. What is a total number of permutations of a sequence of size n? **[4 marks]**

4.C List three properties of free trees. **[6 marks]**

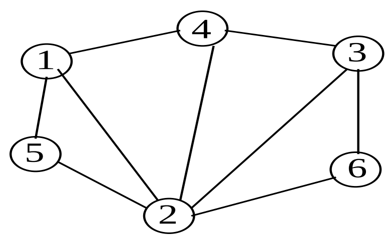
Question 5

5.A Consider the following directed graph G_1:



1. Explain why graph G_1 isn't strongly connected. **[2 marks]**
2. Propose one edge to make G_1 strongly connected. **[2 marks]**
3. What are in-degree and out-degree of node 1. **[2 marks]**
4. What is the shortest directed path from node 1 to 6? **[2 marks]**
5. What is the longest simple path in G_1? **[2 marks]**

5.B Consider the following (undirected) graph G_2:



1. Draw any spanning tree in G_2. **[5 marks]**
2. How many edges are missing to make G_2 complete? **[2 marks]**
3. What is a diameter of G_2? **[2 marks]**
4. What is the minimum and the maximum degree of G_2. **[2 marks]**
5. What is the longest simple path between nodes 1 and 6? **[2 marks]**
6. Is G_2 planar? **[2 marks]**

Answers to question 1

1.A The values of $11^0, 11^1, 11^2, 11^3, 11^4$ form initial consecutive rows in *Pascal's triangle*, i.e. $1, 11, 121, 1331, 14641$. This happens since the value 11^k can be expressed as $(10+1)^k = \sum_{i=0}^{k} \binom{k}{i} 10^i 1^{k-i}$ and the value $\binom{k}{i}$ can be represented as one digit, for any $i \leq k \leq 4$. Unfortunately this simple rule doesn't apply for all $k \geq 5$, since e.g. $\binom{5}{2} = 10$ so it cannot be represented as one digit.

1.B We know that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Thus we have to answer for which values of $0 \leq k \leq n$ the following holds

$$\frac{n!}{(k-1)!(n-k+1)!} \leq \frac{n!}{k!(n-k)!}.$$

Dividing both sides of above inequality by $\frac{n!}{(k-1)!(n-k)!}$ we get

$$\frac{1}{n-k+1} \leq \frac{1}{k},$$

which is equivalent to

$$k \leq n-k+1.$$

And this holds when $2k \leq n+1$, and finally $k \leq \frac{n+1}{2}$. Using that fact we conclude that value $\binom{n}{k}$ is the largest for $k = \lfloor \frac{n+1}{2} \rfloor$.

Answers to question 2

2.A The prove that property (*) $T(n) = \frac{3^n-1}{2}$ holds is performed in two steps.

1. (Basis) Initially we prove that property (*) holds for small integer $n_0 = 1$. And indeed $T(1) = 1$ from the definition of T, and $\frac{3^{n_0}-1}{2} = \frac{3-1}{2} = 1$.
2. (Inductive step) In what follows we assume that property (*) holds for all integers from range $n_0, \ldots, n-1$. We prove now, using the definition of T and inductive assumption that (*) holds for value $n-1$, that property (*) holds also for value n. And indeed

$$T(n) = 3T(n-1) + 1 = 3 \cdot \frac{3^{n-1}-1}{2} + 1 = \frac{3^n - 3 + 2}{2} = \frac{3^n - 1}{2}.$$

Using induction we proved that property (*) holds for all integers.

2.B We use the *linearity property* of a summation and known fact that the sum of initial n integers is $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. Thus

$$\sum_{k=1}^{n}(4k+2) = 4 \cdot \sum_{k=1}^{n} k + \sum_{k=1}^{n} 2 = 4 \cdot \frac{n(n+1)}{2} + n \cdot 2 = 2n \cdot (n+1) + 2n = 2n(n+2).$$

Answers to question 3

3.A

	cost	*#times*
. . .		
$min \leftarrow max \leftarrow A[1]$;	c_1	1
for $i \leftarrow 2$ **to** n **do**	c_2	n
if $min > A[i]$	c_3	$n-1$
then $min \leftarrow A[i]$	c_4	$\leq n-1$
else if $max < A[i]$ **then** $max \leftarrow A[i]$;	c_5	$\leq n-1$
$\triangleright$ *min* contains a minimum value and		
$\triangleright$ *max* contains a maximum value.		
. . .		

The total cost of the solutuion is bounded by:

$$(c_2 + c_3 + c_4 + c_5)n + c_1 - c_3 - c_4 - c_5 = O(n).$$

3.B (i) $n^{\frac{1999}{2000}} < \frac{n}{\log n}$, (ii) $\log(n^{2000}) < n$, (iii) $n^{\log n} > 2^{2000 \log n}$.

3.C $\log^*(256) = 4$, since $2^{2^2} = 16$ and $2^{2^{2^2}} > 256$.

3.D $2000n > n\log(n)$ for all realistic values of n, since inequality holds for all $n < 2^{2000}$ which is much larger then the number of atoms in the solar system.

Answers to question 4

4.A We start with prove that $5n^2 + 2 = O(n^2)$. We have to show that there is a constant $c > 0$ and integer n_0, s.t. $0 \le 5n^2 + 2 \le cn^2$, for all integer $n \ge n_0$. The inequality holds when $n^2(c-5) \ge 2$ and equivalently $n \ge \sqrt{\frac{2}{c-5}}$. If we take $c = 7$ the inequality holds for all $n \ge 1$, thus we can take $n_0 = 1$.

Later we prove that $5n^2 + 2 = \Omega(n^2)$. We have to show that there is a constant $c > 0$ and integer n_0, s.t. $0 \le cn^2 \le 5n^2 + 2$, for all integer $n \ge n_0$. And indeed if we take $c = 1$ we get $n^2 \le 5n^2 + 2$, which is true for any integer $n \ge 1$. Thus in this case $c = 1$ and $n_0 = 1$.

Finally since $5n^2 + 2 = O(n^2)$ and $5n^2 + 2 = \Omega(n^2)$ we get $5n^2 + 2 = \Theta(n^2)$.

4.B $A\ B\ C$, $A\ C\ B$, $B\ A\ C$, $B\ C\ A$, $C\ A\ B$, and $C\ B\ A$. In general, the number of all permutations of a sequence of size n is equal to $n! = 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n$ (n factorial).

4.C A free tree $T = (V, E)$ has the following properties:

1. T is connected, and $|E| = |V| - 1$,
2. T is connected, and has no cycle,
3. there is a uniqe path in T between any two nodes.

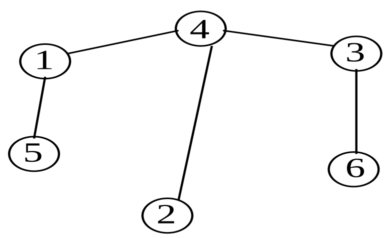
Answers to question 5

5.A

1. Out-degree of node 6 is 0,
2. any out-going edge from node 6, e.g. $6 \to 3$,
3. in-degree is 2, and out-degree is 1,
4. there are two shortest paths: $1 \to 4 \to 3 \to 6$ and $1 \to 4 \to 2 \to 6$,
5. the longest directed simple path has length 4, e.g. $1 \to 4 \to 3 \to 2 \to 6$.

5.B

1. Example of a spanning tree:



2. $\frac{6 \cdot 5}{2} - 9 = 6$, missing edges: $1-2, 1-6, 3-5, 4-5, 4-6, 5-6$,
3. the diameter of G_2 is 2, any node can be reached via node 2
4. the minimum degree is 2 (nodes 5 and 6) and the maximum degree is 5 (node 2),
5. the longest simple path between nodes 1 and 6 is of length 5, see $1-5-2-4-3-6$, and it cannot be longer since we would have a cycle.
6. G_2 is planar since it can be placed on a plane, such that its edges do not cross each other.