Instruction to candidates
(time allowed 2 hours)

- candidates will be assessed on their best four answers,
- if you attempt to answer more than the required number of questions, the marks awarded for the excess questions will be discarded (starting with your lowest mark),
- all logarithms are to the base 2.

Question 1

1.A Prove that:

$$19 < \frac{\binom{2000}{100}}{\binom{2000}{99}} < 20$$

[10 marks]

Is it true that:

$$\binom{2000}{100} < \binom{2000}{1900} \quad ?$$

Jutify your answer. **[5 marks]**

1.B For which values of $0 \leq k \leq n$ is

$$\binom{n}{k-1} \geq \binom{n}{k} \quad ?$$

Which value among $\binom{n}{0}, \binom{n}{1}, \binom{n}{2}, \ldots, \binom{n}{n-1}, \binom{n}{n}$ is the largest? **[10 marks]**

Question 2

2.A Prove, using mathematical induction, that, for $n \geq 1$ $T(n) = \frac{4^n - 1}{3}$, where value $T(n)$ is defined by the recurrence:

$$T(n) = \begin{cases} 1 & \text{for } \; n = 1, \\ 4T(n-1) + 1 & \text{for } \; n > 1, \end{cases}$$

[15 marks]

2.B What is the value of:

$$\sum_{k=1}^{n} (3k + 5) \quad ?$$

Justify your answer. **[10 marks]**

Question 3

3.A Design and write a pseudocode finding positions of a *minimum* and a *maximum* element in array $A[1, \ldots, n]$ of real numbers. The pseudocode should report both extremal values. What is the time complexity of your solution? **[15 marks]**

3.B State, in each of the three cases, which function is larger for almost all n (i.e. which function has higher order):

(i)

$$n^{\frac{2000}{1999}} \qquad n \log(n)$$

(ii)

$$\log^{2000}(n) \qquad n$$

(iii)

$$n^{\log(n)} \qquad 2^{2000 \log(n)}$$

[6 marks]

3.C What is the value of

$$\log^*(2000) \quad ?$$

[2 marks]

3.D Which exact complexity is better for realistic values of parameter n,

$$2000 \qquad \textit{or} \qquad \log(n)?$$

Briefly justify your answer. **[2 marks]**

Question 4

4.A Prove that

$$3n^2 + 2 = O(n^2).$$

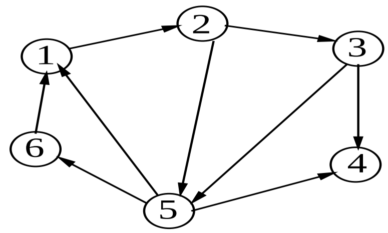
[15 marks]

4.B List all 2-element combinations of the set $\{w, x, y, z\}$. **[4 marks]**

4.C List three properties of free trees. **[6 marks]**

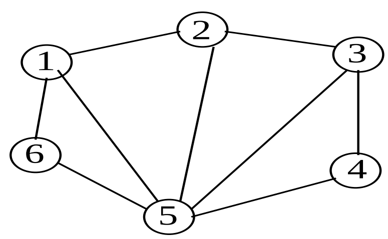
Question 5

5.A Consider the following directed graph G_1:



1. Explain why graph G_1 isn't strongly connected. **[2 marks]**
2. Propose one edge to make G_1 strongly connected. **[2 marks]**
3. What are the in-degree and out-degree of node 5? **[2 marks]**
4. What is the shortest directed path from node 1 to 4? **[2 marks]**
5. What is the longest simple path in G_1? **[2 marks]**

5.B Consider the following (undirected) graph G_2:



1. Draw any spanning tree in G_2. **[5 marks]**
2. How many edges are missing to make G_2 complete? **[2 marks]**
3. What is a diameter of G_2? **[2 marks]**
4. What is the minimum and the maximum degree of G_2? **[2 marks]**
5. What is the longest simple path between nodes 1 and 6? **[2 marks]**
6. Is G_2 planar? **[2 marks]**

Answers to question 1

1.A We use the fact that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. And indeed:

$$\frac{\binom{2000}{100}}{\binom{2000}{99}} = \frac{\frac{2000!}{100! \cdot 1900!}}{\frac{2000!}{99! \cdot 1901!}} = \frac{2000! \cdot 99! \cdot 1901!}{100! \cdot 1900! \cdot 2000!} = \frac{1901}{100} = 19.01$$

The answer to the second question is NO. We use the fact that $\binom{n}{k} = \binom{n}{n-k}$, where $n = 2000$ and $k = 100$.

1.B We know that $\binom{n}{k} = \frac{n!}{k!(n-k)!}$. Thus we have to answer for which values of $0 \leq k \leq n$ the following holds

$$\frac{n!}{(k-1)!(n-k+1)!} \leq \frac{n!}{k!(n-k)!}.$$

Dividing both sides of above inequality by $\frac{n!}{(k-1)!(n-k)!}$ we get

$$\frac{1}{n-k+1} \leq \frac{1}{k},$$

which is equivalent to

$$k \leq n - k + 1.$$

And this holds when $2k \leq n + 1$, and finally $k \leq \frac{n+1}{2}$. Using that fact we conclude that value $\binom{n}{k}$ is the largest for $k = \lfloor \frac{n+1}{2} \rfloor$.

Answers to question 2

2.A The proof that property (*) $T(n) = \frac{4^n-1}{3}$ holds is performed in two steps.

1. (Basis) Initially we prove that property (*) holds for small integer $n_0 = 1$. And indeed $T(1) = 1$ from the definition of T, and $\frac{4^{n_0}-1}{3} = \frac{4-1}{3} = 1$.

2. (Inductive step) In what follows we assume that property (*) holds for all integers from range $n_0, \ldots, n-1$. We prove now, using the definition of T and inductive assumption that (*) holds for value $n-1$, that property (*) holds also for value n. And indeed

$$T(n) = 4T(n-1) + 1 = 4 \cdot \frac{4^{n-1} - 1}{3} + 1 = \frac{4^n - 4 + 3}{3} = \frac{4^n - 1}{3}.$$

Using induction we proved that property (*) holds for all integers ≥ 1.

2.B We use the *linearity property* of a summation and the known fact that the sum of the initial n integers is $\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$. Thus

$$\sum_{k=1}^{n}(3k+5) = 3 \cdot \sum_{k=1}^{n} k + \sum_{k=1}^{n} 5 = 3 \cdot \frac{n(n+1)}{2} + n \cdot 5 = \frac{3}{2}n \cdot (n+1) + 5n.$$

Answers to question 3

3.A

	cost	#times
...		
$min\_pos \leftarrow max\_pos \leftarrow 1;$	c_1	1
for $i \leftarrow 2$ **to** n **do**	c_2	n
if $A[min\_pos] > A[i]$	c_3	$n-1$
then $min\_pos \leftarrow i$	c_4	$\leq n-1$
else if $A[max\_pos] < A[i]$ **then** $max\_pos \leftarrow i;$	c_5	$\leq n-1$
$\triangleright$ $A[min\_pos]$ contains a minimum value and		
$\triangleright$ $A[max\_pos]$ contains a maximum value.		
...		

The total cost of the solutuion is bounded by:

$$(c_2 + c_3 + c_4 + c_5)n + c_1 - c_3 - c_4 - c_5 = O(n).$$

3.B (i) $n^{\frac{2000}{1999}} > n\log(n)$, (ii) $\log^{2000}(n) < n$, (iii) $n^{\log n} > 2^{2000\log n}$.

3.C $\log^*(200) = 4$, since $2^{2^2} = 16 < 2000$ and $2^{2^{2^2}} \geq 2000$.

3.D $2000 > \log(n)$ for all realistic values of n, since inequality holds for all $n < 2^{2000}$ which is much larger then the number of atoms in the solar system.

Answers to question 4

4.A We have to show that there is a constant $c > 0$ and integer n_0, s.t. $0 \leq 3n^2 + 2 \leq cn^2$, for all integer $n \geq n_0$. The inequality holds when $n^2(c-3) \geq 2$ and equivalently $n \geq \sqrt{\frac{2}{c-3}}$. If we take $c = 5$ the inequality holds for all $n \geq 1$, thus we can take $n_0 = 1$.

4.B $\{w, x\}$, $\{w, y\}$, $\{w, z\}$, $\{x, y\}$, $\{x, z\}$, and $\{y, z\}$.

4.C A free tree $T = (V, E)$ has the following properties:

1. T is connected, and $|E| = |V| - 1$,
2. T is connected, and has no cycle,
3. there is a uniqe path in T between any two nodes.

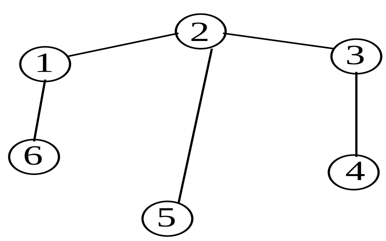
Answers to question 5

5.A

1. Out-degree of node 4 is 0,
2. any out-going edge from node 4, e.g. $6 \to 3$,
3. in-degree is 2, and out-degree is 3,
4. there are two shortest paths: $1 \to 2 \to 3 \to 4$ and $1 \to 2 \to 5 \to 4$,
5. the longest directed simple path has length 5, e.g. $6 \to 1 \to 2 \to 3 \to 5 \to 4$.

5.B

1. Example of a spanning tree:



2. $\frac{6 \cdot 5}{2} - 9 = 6$, missing edges: $1-2, 1-6, 3-5, 4-5, 4-6, 5-6$,
3. the diameter of G_2 is 2, any node can be reached via node 5
4. the minimum degree is 2 (nodes 4 and 6) and the maximum degree is 5 (node 5),
5. the longest simple path between nodes 1 and 6 is of length 5, see $1-2-3-4-5-6$, and it cannot be longer since we would have a cycle.
6. G_2 is planar since it can be placed on a plane, such that its edges do not cross each other.