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THE UNIVERSITY
of LIVERPOOL

SEPTEMBER 2002 EXAMINATIONS

Bachelor of Arts: Year 1
Bachelor of Arts: Year 2
Bachelor of Science: Year 1
Bachelor of Science: Year 2
Master of Mathematics: Year 2

Algorithmic Foundations

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Candidates will be assessed on their best **four** answers.

If you attempt to answer more questions than the required number of questions (in any section), the marks awarded for the excess questions will be discarded (starting with your lowest mark).

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Question 1

1.A Prove or disprove using a truth table that:

$$(P \text{ and } Q) => R$$

is logically equivalent to:

$$(P => R) \text{ or } (\text{not}Q)$$

[10 marks]

1.B Show, by giving a proof by contradiction, that if 100 balls are placed in 9 boxes, some box contains 12 or more balls.

[8 marks]

1.C State the definition of the *Cartesian product* $X \times Y$ of two sets X and Y. Illustrate the definition of the Cartesian product $X \times Y$, where $X = \{a, b\}$ and $Y = \{1, 2, 3\}$, using a Venn diagram.

[7 marks]

Question 2

2.A Prove, using mathematical induction, that

$$2 + 4 + 6 + ... + 2n = n^2 + n.$$

[10 marks]

2.B Eighteen people have first names John, Paddy, and Mary and last names Jones and Kelly. Show using the pigeonhole principle that at least three persons have exactly the same first and last names.

[10 marks]

2.C How many distinct rearrangements are there of the letters in the word *OUAGADOUGOU*.

[5 marks]

Question 3

3.A Let $\mathbf{R}^*$ be the set of real numbers excluding 0. Consider the relation $Q \subseteq \mathbf{R}^* \times \mathbf{R}^*$, such that for any $x, y \in \mathbf{R}^*$, xQy if and only if $x \cdot y$ (x times y) is a positive (> 0) real number. Prove that Q is an *equivalence relation*.

[15 marks]

3.B The function $f : A \rightarrow B$ is given by $f(x) = 1 + \frac{2}{x}$ where A denotes the set of real numbers excluding 0 and B denotes the set of real numbers excluding 1. Show that f is a bijection and determine its inverse function.

[10 marks]

Question 4

4.A Let $\mathcal{A}_n$ be a nonempty sequence of real numbers $a_1, a_2, a_3, ..., a_n$. We say that sequence $\mathcal{A}_n$ contains a *repetition* if $a_i = a_{i+1}$, for some $1 \leq i < n$. Design and write pseudocode of a procedure deciding whether $\mathcal{A}_n$ is a *repetition* free (i.e., it doesn't contain any repetitions) sequence. What is the complexity of your solution?

[15 marks]

4.B Trace the values of i and j in the following algorithm when $n = 5$.

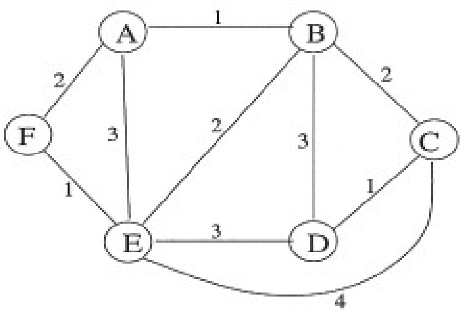
```
begin
    Input n;
    i := 0; j := 1;
    while i < n do
        begin
            i := i + 1; j := j + j;
        end
    Output j;
end
```

What is the output of the algorithm for general n?

[10 marks]

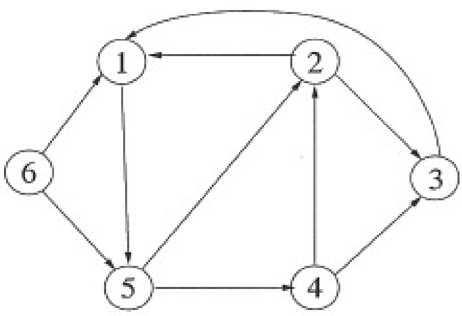
Question 5

5.A The graph G_2 represents a road network connecting a set of six towns where distances are given in miles. Use Prim's algorithm to find a road network of minimal total length connecting all the towns:



[15 marks]

5.B Consider the following directed graph G_1:



1. Explain why graph G_1 isn't strongly connected. **[2 marks]**
2. Propose one additional edge to make G_1 strongly connected. **[2 marks]**
3. What is the in-degree and what is the out-degree of node 5? **[2 marks]**
4. What is the shortest directed path from node 6 to 2? **[2 marks]**
5. Indicate a cycle in G_1 of size 5. **[2 marks]**