PAPER CODE NO. COMP108	EXAMINER : S. Ravindran DEPARTMENT : Computer Science Tel. No. 43670

THE UNIVERSITY
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MAY 2004 EXAMINATIONS

Bachelor of Arts : Year 1
Bachelor of Science : Year 1
Master of Mathematics : Year 2

ALGORITHMIC FOUNDATIONS

TIME ALLOWED : Two Hours

INSTRUCTIONS TO CANDIDATES

Candidates will be assessed on their best **four** answers.

If you attempt to answer more than the required number of questions (in any section), the marks awarded for the excess questions will be discarded (starting with your lowest mark).

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Question 1

1.A Prove the following statement using mathematical induction.

$$1 + 2 + \ldots + n = n(n+1) / 2$$ [10 Marks]

1.B Give, using *big oh* notation, the running time of the following loop as a function of n.

```
for i = 1  to n do
    for j = 1  to i do
        s := s + 1
```

[5 Marks]

1.C Briefly explain why *binary search* takes at most $O(\log n)$ steps to find an element in a sorted array of length n. Assume that $n = 2^k$, for some integer k. [10 Marks]

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Question 2

2.A Assume that n numbers are stored in locations A[1], A[2], …, A[n], where A is an array. We can sort the numbers as follows.

for $i = n$ downto 2 do
  Select the largest among A[i], A[i-1], …, A[1] and swap it with A[i]

In the i^{th} pass, the algorithm selects the largest element by

largenumber := A[i];
for $j = i - 1$ *downto* 1 *do*
    if (A[j] $\geq$ *largenumber*) then *largenumber* := A[j] ;

(i) Prove that this sorting algorithm is not stable.

(ii) How would you modify this algorithm to a stable sorting algorithm.

[10 Marks]

2.B The Quicksort algorithm uses the Divide-and-Conquer technique as follows.

Quicksort (A, p, r)
  If $p \neq r$ then
  begin
    Step 1: Partition the array A[p .. r] into two nonempty sub arrays A[p .. q] and A[q+1 .. r] such that each element of A[p .. q] is less than or equal to each element of A[q+1 .. r].

    Step 2: Recursively sort A[p .. q] (i.e. *Quicksort* (A, p, q))

    Step 3: Recursively sort A[q+1 .. r] (i.e. *Quicksort* (A, q+1, r))
  end.

To sort an entire array A of length n, the initial call is *Quicksort*(A, 1, n).

(i) Assume that Step 1 partitions the array A[1 .. n] into sub arrays A[1 .. i] and A[i+1.. .. n] in O(n) time. Give a recursive equation for the running time of the Quicksort algorithm.

(ii) Using the recursive equation you described in (i), briefly explain the best-case and worst-case behaviour of the Quicksort algorithm. [15 Marks]

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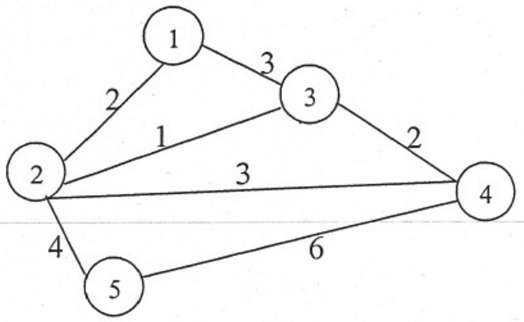
Question 3

3.A Using suitable diagrams, show how a binary search tree would be built for the following sequence of numbers.

5, 8, 10, 7, 3, 9, 1 [5 Marks]

3.B Using a diagram explain that searching for an element may take O(n) steps in an n-node binary search tree. [5 Marks]

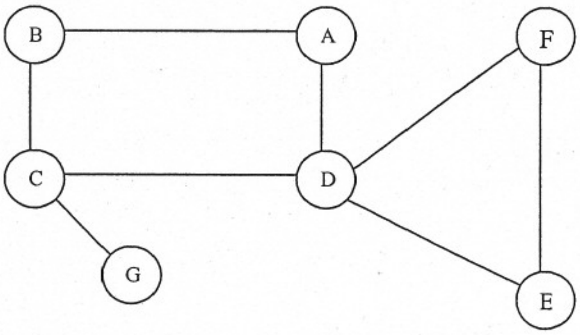
3.C In the following graph the vertices and the edges represent the cities and the direct link between the cities (if any) respectively. A cost of building a direct link is shown as the weight of an edge. We would like to minimize the construction cost and ensure that a path exists between every pair of cities. Removing some links can do this. Using a greedy algorithm, for example Prim's algorithm, to find a minimum weight spanning tree, remove those links.



[10 Marks]

3.D Given a weighted undirected graph $G = (V, E)$, let W be a proper subset of its vertices. Also let e denote the edge of smallest weight with one end in W and the other in $(V - W)$. Show that there exists a minimum weight-spanning tree of G, which contains e. [5 Marks]

Question 4



4.A Give an adjacency matrix representation of the above graph. [5 Marks]

4.B For each vertex v of the above graph compute its position in the order in which the vertices were visited in a Depth First Search ($DFI(v)$), and construct a spanning tree using the edges traversed in the Depth First Search. [10 Marks]

4.C Using the order in which each vertex w was visited in Depth First Search and the spanning tree T, describe an algorithm to find articulation points. [10 Marks]

Question 5

5.A Let D_1 and D_2 denote two decision problems. What does it mean to say that D_1 is polynomially transformable to D_2 (written $D_1 \leq_P D_2$)? [5 Marks]

5.B Prove that, if $D_1 \leq_P D_2$ and D_2 can be solved by a polynomial time algorithm, then D_1 can be solved by a polynomial time algorithm. [5 Marks]

5.C How would you define the set of NP-complete problems? [5 Marks]

5.D (i) Outline the steps for proving that a decision problem D is an NP-complete problem.

(ii)Show that this proof will satisfy the definition you have given in 5.C.
[10 Marks]