MATH545001

This question paper consists of 8 printed pages, each of which is identified by the reference **MATH545001**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH5450

(June 2006)

Polymeric Fluids

Time allowed: **3 hours**

Answer FIVE of the SEVEN questions.

All questions carry equal marks.

1. **(a)** In the power-law fluid model the shear viscosity is equal to $\mu(\dot{\gamma}) = K|\dot{\gamma}|^{n-1}$, where K and n are positive constants. Explain what is meant by the terms *shear thinning* and *shear thickening* and state the range of values of n for which the power law fluid is shear-thinning or shear-thickening.

(b) A cylindrical rod of radius a moves at velocity U along the axis of a cylinder of radius b. The gap between the rod and the cylinder is filled with a power-law fluid. Write down the equation of axial momentum conservation on the assumption that inertia can be neglected and the extra stress, $\boldsymbol{\sigma}$, is a function of r only, and deduce that if the pressure gradient is zero along the pipe then

$$\frac{1}{r}\frac{d}{dr}\left(r\sigma_{rz}\right) = 0.$$

Show that the fluid velocity in the gap between the rod and cylinder is given by

$$u = U\frac{r^{\frac{n-1}{n}} - b^{\frac{n-1}{n}}}{a^{\frac{n-1}{n}} - b^{\frac{n-1}{n}}} \qquad \text{for } n \neq 1.$$

What is the fluid velocity for $n = 1$?

Show that the drag force per unit length acting on rod is equal to

$$F = 2\pi a\sigma_{rz}(a),$$

and find its value when $n = 0.5$.

2. **(a)** Using the appropriate formulae for cylindrical polar coordinates write down the velocity gradient for a flow in which the fluid velocity is given in cylindrical polar coordinates (r, θ, z) by $\mathbf{u} = (0, v(r), 0)$. Show that the $r\theta$ component of the strain-rate tensor, $\mathbf{E}$,

$$E_{r\theta} = \frac{1}{2}\dot{\gamma} = \frac{r}{2}\frac{d}{dr}\left(\frac{v}{r}\right),$$

where $\dot{\gamma}$ is the local shear-rate. Define the shear viscosity, $\mu(\dot{\gamma})$, and first and second normal stress differences, $N_1(\dot{\gamma})$ and $N_2(\dot{\gamma})$, in terms of the components of the extra stress tensor $\boldsymbol{\sigma}$.

(b) Write down the components of the momentum equation for such a flow on the assumption that fluid inertia is negligible and the gravitational acceleration, $\mathbf{g} = (0, 0, -g)$. (You may assume that $\sigma_{rz} = \sigma_{\theta z} = 0$). Show that this leads to the following equations

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\mu(\dot{\gamma})\dot{\gamma}\right) = 0,$$
$$\frac{\partial}{\partial r}\left(-p + \sigma_{zz} + N_2\right) = \frac{N_1}{r},$$
$$\frac{\partial}{\partial z}\left(-p + \sigma_{zz}\right) = \rho g.$$

(c) A vertical rod of radius a rotates at an angular velocity $\Omega > 0$ in a shear-thinning polymeric fluid in which

$$\mu(\dot{\gamma}) = K|\dot{\gamma}|^{-1/3}, \qquad N_1(\dot{\gamma}) = A\dot{\gamma}^2, \qquad N_2(\dot{\gamma}) = 0,$$

where K and A are both positive constants. Show that the fluid velocity $v(r)$ is given by

$$v(r) = \Omega\frac{a^3}{r^2}$$

and hence find $\dot{\gamma}$.

If the top surface is open to the atmosphere, show that the position of this surface is given by

$$h(r) = h_\infty + \frac{3A\Omega^2 a^6}{8\rho g r^6},$$

where h_∞ is the height for $r \to \infty$.

3. **(a)** Write down an expression for the stress in a linear viscoelastic fluid of relaxation modulus $G(t)$. By considering the stress generated by a shear-rate,

$$\dot{\gamma} = \frac{d}{dt}\left(\epsilon \exp(i\omega t)\right),$$

define the complex modulus G^* and explain the significance of the real and imaginary parts of G^*.

(b) The shear stress, $\sigma(t)$ in a linear Maxwell fluid is related to the shear-rate $\dot{\gamma}$ by

$$\tau \frac{d\sigma}{dt} + \sigma = \mu \dot{\gamma}.$$

Show that this is a linear viscoelastic fluid and find its relaxation modulus, $G(t)$. Find the complex modulus of this fluid and show that the loss and storage moduli are given respectively by

$$G' = \frac{\mu \omega^2 \tau}{1+\omega^2\tau^2}, \qquad G'' = \frac{\mu\omega}{1+\omega^2\tau^2}.$$

Find the frequency ω at which $G' = G''$.

(c) A fluid satisfying the linear Maxwell model is subjected to the following shear flow

$$\dot{\gamma} = \begin{cases} & \text{for } t < 0, \\ a \sin \omega t & \text{for } t \geq 0. \end{cases}$$

Find the shear stress for $t > 0$.

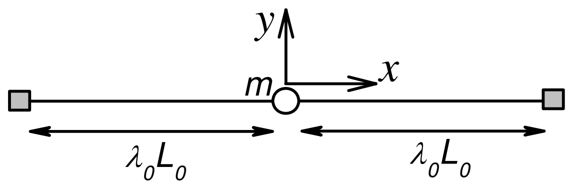
4. The expression for the total stress in a rubber is

$$\boldsymbol{\tau} = G\mathbf{F}\cdot\mathbf{F}^T - \beta\mathbf{I}.$$

where $\mathbf{F}$ is the deformation gradient tensor, G is the shear modulus and β is an isotropic contribution to the pressure.

(a) What is the deformation gradient $\mathbf{F}$ and stress $\boldsymbol{\tau}$ for a volume-conserving uniaxial extension by a ratio λ in the x-direction? A piece of rubber, of initial cross sectional area A_0, is stretched by a ratio λ. If the sides of the rubber are exposed to the atmosphere, so that $\tau_{yy} = \tau_{zz} = -p_{atm}$, show that the force required to achieve the stretch is

$$f = GA_0 \left(\lambda - \frac{1}{\lambda^2}\right).$$



(b) Two light, thin pieces of rubber, of initial length L_0 and initial cross sectional area A_0 are attached by one end to a mass m and stretched to λ_0 times their initial length between clamps a distance $2\lambda_0 L_0$ apart, as shown in the above diagram. Assume the only force on the mass is due to the rubber.

(i) Show that, for small displacements $x \ll \lambda_0 L$ parallel to the rubber direction, the force on the mass in this direction is

$$F_x = -2\frac{GA_0}{L_0}\left(1 + \frac{2}{\lambda_0^3}\right)x.$$

 QUESTION 4 CONTINUED...

(ii) Show that, for small displacements $y \ll \lambda_0 L$ perpendicular to the rubber direction, the force on the mass in this direction is

$$F_y = -2\frac{GA_0}{L_0}\left(1 - \frac{1}{\lambda_0^3}\right)y.$$

(iii) Hence, using Newton's second law $\mathbf{F} = m\mathbf{a}$ for the mass, obtain the ratio ω_x/ω_y of frequencies of small amplitude oscillation of the mass in the x and y directions, as a function of λ_0. What is special about ω_x/ω_y when $\lambda_0^3 = 2$ and $\lambda_0^3 = \frac{11}{8}$?

5. The Langevin equation with intertia for the velocity v of a particle of mass m moving with friction constant ζ is

$$m\frac{dv}{dt} + \zeta v = f(t),$$

where $\langle f(t) f(t')\rangle = 2k_{\mathrm{B}}T\zeta\delta(t - t')$. k_{B} is the Boltzmann constant and T is the temperature.

(a) By use of a suitable integrating factor, or otherwise, show that a solution of this equation is

$$v(t) = \frac{1}{m}\int_{-\infty}^{t} ds\, f(s)\exp\left(\frac{s-t}{\tau}\right),$$

and hence that

$$\langle v(t)\, v(t')\rangle = \frac{1}{m^2}\int_{-\infty}^{t} ds\int_{-\infty}^{t'} ds'\ \langle f(s)\, f(s')\rangle \exp\left(\frac{s + s' - t - t'}{\tau}\right),$$

where $\tau = \frac{m}{\zeta}$.

(b) Hence show that, for $t \leq t'$,

$$\langle v(t)\, v(t')\rangle = \frac{k_{\mathrm{B}}T}{m}\exp\left(-\frac{(t'-t)}{\tau}\right).$$

What is the average kinetic energy, $\frac{1}{2}m\left\langle [v(t)]^2\right\rangle$?

(c) Given that, in general,

$$\langle v(t)\, v(t')\rangle = \frac{k_{\mathrm{B}}T}{m}\exp\left(-\frac{|t'-t|}{\tau}\right)$$

and that the position of the particle is $x(t) = \int_0^t ds\, v(s)$ when $x(0) = 0$, show that

$$\left\langle [x(t)]^2\right\rangle = \frac{2k_{\mathrm{B}}T}{\zeta}\left(t + \tau\left(\exp\left(-\frac{t}{\tau}\right) - 1\right)\right).$$

From the limiting behaviour of this as t becomes large, obtain the diffusion constant for the motion.

Hint: $\int_0^t ds \int_0^t ds' f(|s - s'|) = 2\int_0^t ds \int_0^s ds' f(s - s')$.

6. The Rouse equation for a polymer chain comprising beads with friction constant ζ connected with springs of spring constant k is

$$\zeta\left(\frac{\partial \mathbf{r}_s}{\partial t} - \mathbf{v}\left(\mathbf{r}_s\right)\right) = k\frac{\partial^2 \mathbf{r}_s}{\partial s^2} + \mathbf{f}_s,\ s = 0..N,$$

with boundary conditions

$$\left.\frac{\partial \mathbf{r}_s}{\partial s}\right|_{s=0} = 0 \text{ and } \left.\frac{\partial \mathbf{r}_s}{\partial s}\right|_{s=N} = 0.$$

(a) In terms of the forces acting on a bead, briefly discuss the origin of the term, $\frac{\partial^2 \mathbf{r}_s}{\partial s^2}$.

(b) Ignoring the terms due to velocity, $\mathbf{v}\left(\mathbf{r}_s\right)$, and random force, $\mathbf{f}_s$, show that the relaxation time of the pth normal mode, $\mathbf{r}_s = \mathbf{X}_p \cos\left(\frac{\pi p s}{N}\right)$, is

$$\tau_p = \frac{\tau_1}{p^2},$$

where $\tau_1 = \frac{N^2\zeta}{\pi^2 k}$.

(c) Given that this leads to a time-dependent modulus of form

$$G\left(t\right) = G_0 \sum_{p=1}^{\infty} \exp\left(-\frac{p^2 t}{\tau_1}\right),$$

use $G^* = G' + iG'' = \int_0^\infty i\omega G\left(s\right)\exp\left(-i\omega s\right) ds$ to show that

$$\begin{aligned} G' &= G_0 \sum_{p=1}^{\infty} \frac{\omega^2\tau_1^2}{p^4 + \omega^2\tau_1^2}, \\ G'' &= G_0 \sum_{p=1}^{\infty} \frac{p^2\omega\tau_1}{p^4 + \omega^2\tau_1^2}, \end{aligned}$$

and obtain approximations of the form $G'' = c\omega^\alpha$ for $\omega\tau_1 \ll 1$ and (by approximating the sum as an integral) for $\omega\tau_1 \gg 1$. Hence sketch a graph of $\log G''$ versus $\log \omega$.

You may use the results:

$$\sum_1^\infty p^{-2} = \frac{\pi^2}{6},$$

$$\int_0^\infty \frac{x^2}{1+x^4} dx = \frac{\pi}{2\sqrt{2}}.$$

7. In the Oldroyd B model the extra stress $\boldsymbol{\sigma}$ is given by a sum of a Newtonian fluid stress and an Upper Convected Maxwell element:

$$\boldsymbol{\sigma} = G\mathbf{A} + \mu\left(\mathbf{K} + \mathbf{K}^T\right),$$

where the structure tensor $\mathbf{A}$ satisfies

$$\frac{d\mathbf{A}}{dt} = \mathbf{K}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{K}^T - \frac{1}{\tau}\left(\mathbf{A} - \mathbf{I}\right).$$

Here $\mathbf{K}$ is the velocity gradient tensor, with components $K_{ij} = \frac{\partial u_i}{\partial x_j}$ and G and τ are both positive constants. μ is the Newtonian viscosity.

(a) Find the tensor $\mathbf{K}$ and hence write down the equations for evolution of the tensor components, A_{xx}, A_{xy} and A_{yy} when an Oldroyd B fluid is subjected to a transient shear flow of the form $\mathbf{u} = (yf(t), 0, 0)$.

By setting $\frac{d\mathbf{A}}{dt}$ to zero, obtain the steady state values of A_{xx}, A_{xy} and A_{yy} when the fluid is subjected to a shear flow at constant rate of $f(t) = g$.

(b) An Oldroyd B fluid is subjected to a constant shear flow $f(t) = g$ for $t < 0$, and at $t = 0$ has reached steady state. For $t > 0$, the shear stess is removed so that $\sigma_{xy} = 0$, but the fluid remains constrained to a flow of form $\mathbf{u} = (yf(t), 0, 0)$.

Using $\sigma_{xy} = 0$, show that, for $t > 0$,

$$f(t) = -\frac{G}{\mu}A_{xy}(t).$$

Hence obtain A_{yy}, A_{xy} and $f(t)$ as a function of t for $t > 0$. Find the recoil strain

$$\gamma_{\mathrm{R}} = -\int_0^\infty f(t)dt.$$

For $t > 0$, find A_{xx} and show that the normal stress $N_1 = \sigma_{xx} - \sigma_{yy}$ is

$$N_1 = 2Gg^2\tau^2\left(\frac{G\tau}{2G\tau+\mu}\exp\left(-2\left[\frac{G}{\mu}+\frac{1}{\tau}\right]t\right) + \frac{G\tau+\mu}{2G\tau+\mu}\exp\left(-\frac{t}{\tau}\right)\right).$$

Formula Sheet

Cartesian coordinates

pressure, p, velocity, $\mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$, velocity gradient, $\mathbf{K}$ with $K_{ij} = \frac{\partial u_i}{\partial x_j}$

$$\nabla p = \frac{\partial p}{\partial x}\mathbf{e}_x + \frac{\partial p}{\partial y}\mathbf{e}_y + \frac{\partial p}{\partial z}\mathbf{e}_z, \qquad \nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

$$\mathbf{K} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\[2ex] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\[2ex] \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{pmatrix} \qquad \nabla \cdot \boldsymbol{\sigma} = \begin{pmatrix} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} + \dfrac{\partial \sigma_{zx}}{\partial z} \\[2ex] \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{zy}}{\partial z} \\[2ex] \dfrac{\partial \sigma_{xz}}{\partial x} + \dfrac{\partial \sigma_{yz}}{\partial y} + \dfrac{\partial \sigma_{zz}}{\partial z} \end{pmatrix}$$

Cylindrical Polar Coordinates

velocity, $\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_z$.

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{\partial p}{\partial z}\mathbf{e}_z, \qquad \nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z},$$

$$\mathbf{K} = \begin{pmatrix} \dfrac{\partial u}{\partial r} & \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r} & \dfrac{\partial u}{\partial z} \\[2ex] \dfrac{\partial v}{\partial r} & \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} + \dfrac{u}{r} & \dfrac{\partial v}{\partial z} \\[2ex] \dfrac{\partial w}{\partial r} & \dfrac{1}{r}\dfrac{\partial w}{\partial \theta} & \dfrac{\partial w}{\partial z} \end{pmatrix}$$

$$\nabla \cdot \boldsymbol{\sigma} = \begin{pmatrix} \dfrac{1}{r}\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{1}{r}\dfrac{\partial \sigma_{\theta r}}{\partial \theta} + \dfrac{\partial \sigma_{zr}}{\partial z} - \dfrac{\sigma_{\theta\theta}}{r} \\[2ex] \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\sigma_{r\theta}\right) + \dfrac{1}{r}\dfrac{\partial \sigma_{\theta\theta}}{\partial \theta} + \dfrac{\partial \sigma_{z\theta}}{\partial z} + \dfrac{\sigma_{\theta r} - \sigma_{r\theta}}{r} \\[2ex] \dfrac{1}{r}\dfrac{\partial}{\partial r}(r\sigma_{rz}) + \dfrac{1}{r}\dfrac{\partial \sigma_{\theta z}}{\partial \theta} + \dfrac{\partial \sigma_{zz}}{\partial z} \end{pmatrix}$$

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Spherical Polar Coordinates

$\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_\phi$

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}\mathbf{e}_\phi,$$

$$\nabla \cdot \mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(v\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial w}{\partial \phi},$$

$$\mathbf{K} = \begin{pmatrix} \dfrac{\partial u}{\partial r} & \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r} & \dfrac{1}{r\sin\theta}\dfrac{\partial u}{\partial \phi} - \dfrac{w}{r} \\ \dfrac{\partial v}{\partial r} & \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} + \dfrac{u}{r} & \dfrac{1}{r\sin\theta}\dfrac{\partial v}{\partial \phi} - \dfrac{w}{r}\cot\theta \\ \dfrac{\partial w}{\partial r} & \dfrac{1}{r}\dfrac{\partial w}{\partial \theta} & \dfrac{1}{r\sin\theta}\dfrac{\partial w}{\partial \phi} + \dfrac{u}{r} + \dfrac{v}{r}\cot\theta \end{pmatrix}$$

$$\nabla \cdot \boldsymbol{\sigma} = \begin{pmatrix} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sigma_{\theta r}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi r}}{\partial \phi} - \frac{\sigma_{\theta\theta}+\sigma_{\phi\phi}}{r} \\ -\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\sigma_{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sigma_{\theta\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\theta}}{\partial \phi} + \frac{\sigma_{\theta r}-\sigma_{r\theta}-\sigma_{\phi\phi}\cot\theta}{r} \\ \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\sigma_{r\phi}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sigma_{\theta\phi}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{\sigma_{\phi r}-\sigma_{r\phi}+\sigma_{\phi\theta}\cot\theta}{r} \end{pmatrix}$$

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