MATH5450M01

This question paper consists of 8 printed pages, each of which is identified by the reference **MATH5450M01**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH5450M

(June 2005)

Polymeric Fluids

Time allowed: **3 hours**

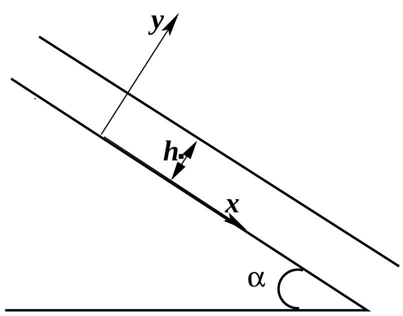
Answer FIVE of the SEVEN questions.

All questions carry equal marks.

1. In the power-law fluid model the shear viscosity is equal to $\mu(\dot{\gamma}) = K|\dot{\gamma}|^{n-1}$, where $\dot{\gamma}$ is the shear-rate and K and n are positive constants.

(a) Explain what is meant by the terms *shear thinning* and *shear thickening* and state the range of values of n for which the power law fluid is shear-thinning or shear-thickening.

(b) A plane inclined at an angle α to the horizontal is coated with a layer of power-law fluid of thickness h.



Defining Cartesian coordinates with x directed down the slope and y perpendicular to the slope and assuming from symmetry that the fluid velocity is of the form $\mathbf{u} = (u(y), 0)$, show that the x and y components of the momentum equation reduce to

$$\frac{\partial p}{\partial x} = \frac{\partial \sigma_{yx}}{\partial y} + \rho g \sin\alpha,$$
$$\frac{\partial p}{\partial y} = -\rho g \cos\alpha,$$

QUESTION 1 CONTINUED...

where ρ is the fluid density and g is the gravitational acceleration. State the boundary conditions that apply on the free surface $y = h$. Hence find the pressure, p, and show that the shear stress σ_{yx} is given by

$$\sigma_{yx} = \rho g (h - y) \sin \alpha.$$

Find the form of the fluid velocity $u(y)$ and show that the velocity at the free surface is proportional to $h^{\frac{n+1}{n}}$.

(c) The fluid in part **(b)** is replaced with a plastic material with yield stress, σ_y. Find the position of the yield surface and the angle α for which the material just flow down the slope. What is the maximum thickness, h, for which the coating will remain fixed for all angles α?

2. A polymeric fluid is contained between two parallel circular disks of radius a that are a distance h apart. The fluid is open to the atmosphere at $r = a$. The upper disk is rotated at angular velocity Ω while the lower disk remains fixed, so that in cylindrical polar coordinates the fluid velocity between the plates is given by

$$\mathbf{u} = \frac{\Omega r z}{h} \hat{\boldsymbol{\theta}},$$

where $\hat{\boldsymbol{\theta}}$ is the unit vector in the angular direction.

(a) Calculate the strain-rate tensor, $\mathbf{E}$ for this flow and show that it corresponds to a shear flow with a shear-rate $\dot{\gamma} = \frac{r\Omega}{h}$.

Identify the *flow*, *gradient* and *vorticity* directions and hence define the normal stress differences N_1 and N_2 in terms of the components of the stress tensor $\boldsymbol{\tau}$.

Show that a normal force equal to

$$F = 2\pi \int_0^a (\tau_{rr} + N_2 + p_{\text{atm}})\, r dr$$

is required to maintain the separation of the plates, where p_{atm} is atmospheric pressure, and show from the radial momentum equation that

$$\frac{\partial \tau_{rr}}{\partial r} = \frac{N_1 + N_2}{r}.$$

(b) For the case $N_1(\dot{\gamma}) = A_1 \dot{\gamma}$ and $N_2(\dot{\gamma}) = A_2 \dot{\gamma}$ where A_1 and A_2 are constants, show that

$$\tau_{rr} = -p_{\text{atm}} + \frac{(A_1 + A_2)\Omega}{h}(r - a).$$

Hence find the force, F.

3. **(a)** The extra stress $\boldsymbol{\sigma}$ in the linear Maxwell model is related to the strain-rate, $\mathbf{E}(t)$ by

$$\tau\frac{\partial\boldsymbol{\sigma}}{\partial t} + \boldsymbol{\sigma} = 2\mu\mathbf{E}(t).$$

where τ and μ are constants.

Show that this may be written in the form

$$\boldsymbol{\sigma} = 2\int_{-\infty}^{t} G(t-t')\mathbf{E}(t')dt', \tag{1}$$

for some suitable choice for the relaxation modulus $G(t)$. Sketch a graph of $G(t)$ and explain the significance of the parameter τ.

Show, using equation (1), that the steady shear viscosity is equal to μ.

(b) Find the form of shear stress $\sigma_{xy}(t)$ generated by the fluid velocity $\mathbf{u} = (y\dot{\gamma}(t), 0, 0)$, where

$$\dot{\gamma} = \begin{cases} \dot{\gamma}_0 & t < 0, \\ -\frac{1}{2}\dot{\gamma}_0 & 0 \leq t \leq T, \\ 0 & t > T, \end{cases}$$

and $\dot{\gamma}_0$ is a positive constant for (i) $t < 0$, (ii) $0 \leq t \leq T$ and (iii) $t > T$. Show that if T is chosen to be equal to a particular value, T_{crit}, then $\sigma_{xy} = 0$ for $t > T$. Sketch graphs of σ_{xy} as a function of time for $T < T_{\text{crit}}$, $T = T_{\text{crit}}$ and $T > T_{\text{crit}}$.

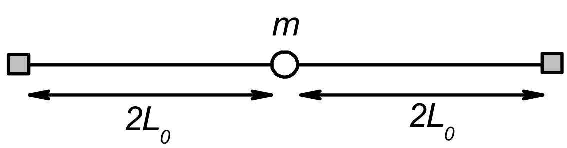
4. The expression for the total stress in a rubber is

$$\boldsymbol{\tau} = G\mathbf{F}\cdot\mathbf{F}^T - \beta\mathbf{I}.$$

where $\mathbf{F}$ is the deformation gradient tensor, G is the shear modulus and β is an isotropic contribution to the pressure.

(a) What is the deformation gradient $\mathbf{F}$ and stress $\boldsymbol{\tau}$ for a volume-conserving uniaxial extension by a ratio λ in the x-direction? A piece of rubber, of initial cross sectional area A_0, is stretched by a ratio λ. If the sides of the rubber are exposed to the atmosphere, so that $\tau_{yy} = \tau_{zz} = -p_{atm}$, show that the force required to achieve the stretch is

$$f = GA_0\left(\lambda - \frac{1}{\lambda^2}\right).$$



(b) Two light, thin pieces of rubber, of initial length L_0 and initial cross sectional area A_0 are attached by one end to a mass m and stretched to twice their initial length between clamps a distance $4L_0$ apart, as shown in the above diagram. Assume the only force on the mass is due to the rubber.

 QUESTION 4 CONTINUED...

(i) For horizontal displacements x from the equilibrium position of the mass, obtain the total force on the mass due to the rubber in terms of x, L_0, G and A_0.

(ii) Show that, for small values of x/L_0, the force on the mass is

$$F = -\frac{5GA_0}{2}\frac{x}{L_0}.$$

(iii) The mass is initially held at the equilibrium position, then fired horizontally with initial velocity V to the right. Given that a thin piece rubber becomes "slack" for $\lambda < 1,$ show that this will occur provided

$$V^2 > \frac{8GA_0L_0}{3m}.$$

Hint: recall that the acceleration, $\frac{dv}{dt} = v\frac{dv}{dx}$.

5. A set of particles are allowed to move in the (x, y) plane, are subjected to a quadratic potential $U = \frac{1}{2}\left(k_x x^2 + k_y y^2\right)$, and placed in a shear flow with shear gradient in the y-direction (and flow in the x-direction) so that the equations of motion of each particle are

$$\begin{aligned} \zeta\left(\frac{dx}{dt} - \dot{\gamma} y\right) &= -k_x x + f_x(t), \\ \zeta\frac{dy}{dt} &= -k_y y + f_y(t). \end{aligned}$$

where $\langle x(t) f_x(t)\rangle = \langle y(t) f_y(t)\rangle = k_{\mathrm{B}}T$ and $\langle x(t) f_y(t)\rangle = \langle y(t) f_x(t)\rangle = 0.$ k_{B} is Boltzmann's constant and T is the temperature.

(a) If $k_x = k_y = k$, show that the variables $Q_{yy}(t) = \langle y^2\rangle$ and $Q_{xy}(t) = \langle xy\rangle$ satisfy the equations

$$\begin{aligned} \frac{dQ_{yy}}{dt} &= -2\frac{k}{\zeta}Q_{yy} + 2\frac{k_{\mathrm{B}}T}{\zeta} \\ \frac{dQ_{xy}}{dt} &= \dot{\gamma}Q_{yy} - 2\frac{k}{\zeta}Q_{xy} \end{aligned}$$

and obtain a similar equation for $Q_{xx}(t) = \langle x^2\rangle$.

(b) Using the set of equations obtained in part (a), find the steady state values of Q_{yy} and Q_{xy}, and show that the steady state value of Q_{xx} is

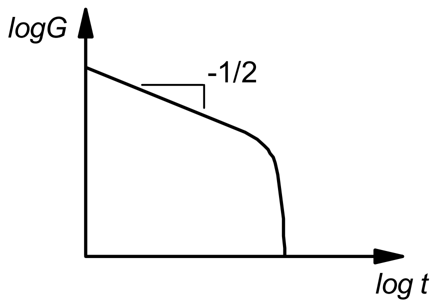
$$Q_{xx} = \frac{k_{\mathrm{B}}T}{k}\left(1 + \frac{\zeta^2\dot{\gamma}^2}{2k^2}\right).$$

(c) The x component of the potential is now switched off, so that $k_x = 0$ and $k_y = k$. Obtain the new dynamical equations for Q_{xx}, Q_{xy} and Q_{yy}. Show that these new equations are satisfied by

$$Q_{yy} = c_1, \quad Q_{xy} = c_2 \quad \text{and} \quad Q_{xx} = 2D_{\text{eff}}t,$$

obtaining the constants c_1 and c_2 and finding the effective diffusion constant D_{eff}.

6. **(a)** The graph below is a sketch of the relaxation modulus $G(t)$ for an unentangled melt. Sketch similar graphs of the relaxation modulus for (i) an entangled melt, and (ii) a rubber, and briefly explain the differences between the graphs.



(b) The reptation contribution to the relaxation modulus for an entangled melt is

$$G(t) = \frac{G_e}{L}\int_0^L p(s,t)\,ds$$

where p is defined for $0 \le s \le L$ so that

$$\frac{\partial p}{\partial t} = D\frac{\partial^2 p}{\partial s^2}$$

and

$$\begin{aligned} p(0,t) &= p(L,t) = 0 \\ p(s,0) &= 1 \end{aligned}$$

Obtain a solution for $p(s,t)$ by separation of variables, and hence show that

$$G(t) = G_e \sum_{n \text{ odd}} \frac{8}{\pi^2 n^2} \exp\left(-\frac{n^2 t}{\tau_d}\right)$$

obtaining an expression for the time τ_d in terms of L and D.

You may use the result $\int_0^\pi \sin(mx)\sin(nx)\,dx = \frac{\pi}{2}\delta_{mn}$.

(c) Obtain the viscosity and recoil after steady shear, R,

$$R = \frac{\dot{\gamma}\int_0^\infty sG(s)\,ds}{\int_0^\infty G(s)\,ds}$$

for the reptation model.
You may use the results $\sum_{p \text{ odd}} p^{-4} = \frac{\pi^4}{96}$, $\sum_{p \text{ odd}} p^{-6} = \frac{\pi^6}{960}$.

7. In the Upper Convected Maxwell model the extra stress $\boldsymbol{\sigma}$ is given by

$$\boldsymbol{\sigma} = G\mathbf{A},$$

where the structure tensor $\mathbf{A}$ satisfies

$$\frac{d\mathbf{A}}{dt} = \mathbf{K} \cdot \mathbf{A} + \mathbf{A} \cdot \mathbf{K}^T - \frac{1}{\tau}\left(\mathbf{A} - \mathbf{I}\right).$$

Here $\mathbf{K}$ is the velocity gradient tensor, with components $K_{ij} = \frac{\partial u_i}{\partial x_j}$ and G and τ are both positive constants.

(a) Write down the equations for evolution of the tensor components, A_{xx}, A_{xy}, A_{yy} and A_{zz} when an Upper Convected Maxwell fluid is subjected to a transient shear flow of the form $\mathbf{u} = (yf(t), 0, 0)$, for $t > 0$.

Deduce that if $\mathbf{A} = \mathbf{I}$ at $t = 0$, the second normal stress difference $N_2 = \sigma_{yy} - \sigma_{zz}$ is equal to zero for $t > 0$ for all functions $f(t)$.

(b) Show that if

$$f(t) = a\exp(at),$$

the shear stress σ_{xy} for $t > 0$ is given by

$$\sigma_{xy} = \frac{a\tau}{a\tau + 1}\Big[\exp(at) - \exp(-t/\tau)\Big].$$

Find the form of the first normal stress difference $N_1 = \sigma_{xx} - \sigma_{yy}$.

Formula Sheet

Cartesian coordinates

pressure, p, velocity, $\mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$, velocity gradient, $\mathbf{K}$ with $K_{ij} = \frac{\partial u_i}{\partial x_j}$

$$\nabla p = \frac{\partial p}{\partial x}\mathbf{e}_x + \frac{\partial p}{\partial y}\mathbf{e}_y + \frac{\partial p}{\partial z}\mathbf{e}_z, \qquad \nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

$$\mathbf{K} = \begin{pmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{pmatrix} \qquad \nabla \cdot \boldsymbol{\sigma} = \begin{pmatrix} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} + \frac{\partial \sigma_{zx}}{\partial z} \\ \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{zy}}{\partial z} \\ \frac{\partial \sigma_{xz}}{\partial x} + \frac{\partial \sigma_{yz}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} \end{pmatrix}$$

Cylindrical Polar Coordinates

velocity, $\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_z$.

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{\partial p}{\partial z}\mathbf{e}_z, \qquad \nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z},$$

$$\mathbf{K} = \begin{pmatrix} \frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial \theta} - \frac{v}{r} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial r} & \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial r} & \frac{1}{r}\frac{\partial w}{\partial \theta} & \frac{\partial w}{\partial z} \end{pmatrix}$$

$$\nabla \cdot \boldsymbol{\sigma} = \begin{pmatrix} \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rr}) + \frac{1}{r}\frac{\partial \sigma_{\theta r}}{\partial \theta} + \frac{\partial \sigma_{zr}}{\partial z} - \frac{\sigma_{\theta\theta}}{r} \\ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{r\theta}\right) + \frac{1}{r}\frac{\partial \sigma_{\theta\theta}}{\partial \theta} + \frac{\partial \sigma_{z\theta}}{\partial z} + \frac{\sigma_{\theta r} - \sigma_{r\theta}}{r} \\ \frac{1}{r}\frac{\partial}{\partial r}(r\sigma_{rz}) + \frac{1}{r}\frac{\partial \sigma_{\theta z}}{\partial \theta} + \frac{\partial \sigma_{zz}}{\partial z} \end{pmatrix}$$

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Spherical Polar Coordinates

$\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_\phi$

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}\mathbf{e}_\phi,$$

$$\nabla\cdot\mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(v\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial w}{\partial\phi},$$

$$\mathbf{K} = \begin{pmatrix} \frac{\partial u}{\partial r} & \frac{1}{r}\frac{\partial u}{\partial\theta} - \frac{v}{r} & \frac{1}{r\sin\theta}\frac{\partial u}{\partial\phi} - \frac{w}{r} \\ \frac{\partial v}{\partial r} & \frac{1}{r}\frac{\partial v}{\partial\theta} + \frac{u}{r} & \frac{1}{r\sin\theta}\frac{\partial v}{\partial\phi} - \frac{w}{r}\cot\theta \\ \frac{\partial w}{\partial r} & \frac{1}{r}\frac{\partial w}{\partial\theta} & \frac{1}{r\sin\theta}\frac{\partial w}{\partial\phi} + \frac{u}{r} + \frac{v}{r}\cot\theta \end{pmatrix}$$

$$\nabla\cdot\boldsymbol{\sigma} = \begin{pmatrix} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sigma_{\theta r}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\phi r}}{\partial\phi} - \frac{\sigma_{\theta\theta}+\sigma_{\phi\phi}}{r} \\ -\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\sigma_{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sigma_{\theta\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\phi\theta}}{\partial\phi} + \frac{\sigma_{\theta r}-\sigma_{r\theta}-\sigma_{\phi\phi}\cot\theta}{r} \\ \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\sigma_{r\phi}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sigma_{\theta\phi}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial\sigma_{\phi\phi}}{\partial\phi} + \frac{\sigma_{\phi r}-\sigma_{r\phi}+\sigma_{\phi\theta}\cot\theta}{r} \end{pmatrix}$$

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