This question paper consists of 3 printed pages, each of which is identified by the reference MATH-543101

Only approved basic scientific calculators may be used.

Examination for the Module MATH-543101
(June 2006)

Nonlinear Dynamics

Time allowed: **2 hours**

Attempt **three** questions. All questions carry equal marks.

Q1 Consider the system of ordinary differential equations

$$\frac{dx}{dt} = y - 2x, \qquad \frac{dy}{dt} = a + x^2 - y,$$

where a is a parameter.

(a) Show that for $a > 1$ there are no equilibria, and for $a < 1$ there are two equilibria. Examine the stability of both fixed points and classify them for all values of $a < 1$.

(b) What kind of bifurcation occurs at $a = 1$? Find the eigenvalues and eigenvectors of the fixed point at $a = 1$ and show that

$$u = \frac{1}{3}(x + y - 3), \qquad v = \frac{1}{3}(2x - y)$$

is a set of normal coordinates for the bifurcation.

(c) In the neighbourhood of $a = 1$, u and v are small, and $\mu = 1 - a$ is $O(u^2)$. The extended centre manifold can be written as a power series in u and μ of the form

$$v = a_1\mu + a_2u^2 + \text{ higher order terms.}$$

Why is there no term proportional to u in the v expansion? Determine the first two coefficients a_1 and a_2 and show that u evolves according to

$$\frac{du}{dt} = b_1\mu + b_2u^2 + \text{ higher order terms,}$$

where the coefficients b_1 and b_2 should also be determined.

Q2 **(a)** Show that the system of differential equations

$$\frac{dx}{dt} = \mu x - y - x^3, \qquad \frac{dy}{dt} = x + \mu y - y^3$$

has a Hopf bifurcation at $\mu = 0$.

(b) For $\mu = 1$, put the equations into polar coordinates r and θ using $x = r\cos\theta$, $y = r\sin\theta$, and hence show that

$$\frac{dr}{dt} = r - r^3(1 - \frac{1}{2}\sin^2 2\theta), \qquad \frac{d\theta}{dt} = 1 + \frac{r^2}{4}\sin 4\theta.$$

Deduce that $\dot{r} > 0$ if $r < 1$ and $\dot{r} < 0$ if $r > \sqrt{2}$. Show also that $r = 0$ is the only fixed point, and hence that there must be a stable limit cycle in $1 \le r \le \sqrt{2}$.

(c) Show that for μ small, a limit cycle exists with $z = x + iy = A\exp i\omega t$, with

$$\omega = 1 + \omega_2 A^2 + \cdots, \quad \mu = \mu_2 A^2 + \cdots, \text{ and } \quad z = A\exp i\omega t + O(A^3).$$

Show that $\omega_2 = 0$, find μ_2 and hence the amplitude A in terms of μ. Is the small amplitude limit cycle stable or unstable? Is the value of A predicted by this weakly nonlinear theory at $\mu = 1$ within the region containing the limit cycle established in part (b)?

Q3 **(a)** Show that the Takens-Bogdanov system

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = \mu_1 + \mu_2 y + x^2 + xy$$

has a double zero eigenvalue at $\mu_1 = \mu_2 = 0$. Write down the Jordan form of the linearised system about $x = y = 0$.

(b) Show that there are no equilibria for $\mu_1 > 0$. How can we be sure there are no limit cycles when $\mu_1 > 0$?

(c) Show that there is a saddle-node bifurcation at $\mu_1 = 0$. Identify which of the two equilibria resulting from this bifurcation is the saddle and which is the node. Show that the node transforms into a focus as $\mu_2 \to \sqrt{-\mu_1}$, and that it then undergoes a Hopf bifurcation on the curve $\mu_2 = \sqrt{-\mu_1}$.

(d) The Hopf bifurcation can be studied by using the transformation

$$u = x + \sqrt{-\mu_1}, \quad v = -\frac{y}{\omega}, \quad \omega^2 = 2\sqrt{-\mu_1}, \quad \hat{\mu} = \mu_2 - \sqrt{-\mu_1}$$

to convert the system into the form

$$\frac{du}{dt} = -\omega v + f(u,v), \quad \frac{dv}{dt} = \hat{\mu} v + \omega u + g(u,v),$$

where f and g are quadratic functions of u and v. Here f is zero and you should find g. Given that at $\hat{\mu} = 0$ the normal form of the Hopf bifurcation is

$$\dot{z} = i\omega z + az|z|^2,$$

where for quadratic f and g,

$$a = \frac{1}{16\omega}\left[f_{uv}(f_{uu} + f_{vv}) - g_{uv}(g_{uu} + g_{vv}) - f_{uu}g_{uu} + f_{vv}g_{vv}\right],$$

establish that the Hopf bifurcation is subcritical. Show that the fixed point $(-\sqrt{-\mu_1}, 0)$ is stable for $\hat{\mu} < 0$ and explain why this implies that an unstable limit cycle exists for $\hat{\mu} < 0$.

(e) How is the limit cycle created by the Hopf bifurcation destroyed as μ_2 is reduced below $\sqrt{-\mu_1}$?

Q4 **(a)** The system of ordinary differential equations

$$\frac{dx}{dt} = \beta x + f(x, y, \mu), \quad \frac{dy}{dt} = -\gamma y + g(x, y, \mu),$$

where f and g are homogeneous quadratic functions of x and y, and μ is a parameter, have a saddle point with β and γ positive at $x = y = 0$. Define the saddle index, δ, and the stable and unstable manifolds of the fixed point $x = y = 0$.

State what is meant by a homoclinic connection between the stable and unstable manifolds. You are given that a homoclinic connection exists at $\mu = \mu_0$. Explain how the limit cycle that can exist for μ close to μ_0 can be analysed in terms of the composition of two maps, one of which is linear. Show that the composite map can be written

$$x_{n+1} = A(\mu - \mu_0) + Bx_n^\delta,$$

where A and B are constants you should define. Discuss the cases $\delta > 1$ and $\delta < 1$.

(b) The system

$$\frac{dx}{dt} = \mu x + y - x^2, \quad \frac{dy}{dt} = -x + 2x^2,$$

has a homoclinic connection at $\mu = \mu_0 \approx 0.135$. Find the fixed points of this system, and identify which is the saddle at $\mu = \mu_0$. Evaluate the saddle index δ at $\mu = \mu_0$, and hence find whether the associated limit cycle is stable or unstable. Is your result consistent with the local behaviour near the other fixed point?