MATH5300-01

This question paper consists of 2 printed pages plus a formula sheet, each of which is identified by the reference MATH5300-01

Only approved basic scientific calculators may be used

Examination for the Module MATH5300
(May 2005)
Applied Financial Modelling

Time allowed: 3 hours

Attempt no more than 5 questions. All questions are of equal value.
Show your working in answers to all questions.

1. Assume that interest is compounded continuously with rate 5%, and that a year has 365 days.

(a) i. If £1000000 is deposited in a bank account on 1 August 2005, how much will be in the account on 2 August 2005?

ii. How much must be deposited in a bank account on 1 August 2005, in order to have £2000000 on 24 August 2006?

(b) The price of a European put option is £1.00. Its expiry date is 1 year from now and its strike price is £5.00. The current price of the stock is £4.00.

i. Use put-call parity to find the current price of a call option on the same stock.

ii. If the price of the stock suddenly increases to £6.00, will the prices of the put and call options increase or decrease? Give a short explanation of your answers.

(c) An investor buys shares in the company PAI at £1.25 on 1 January 2005, receives a dividend of £0.10 per share on 1 July 2005 and sells the shares for £1.31 on 1 July 2006. What is the profit or loss per share, taking into account interest rates?

2. (a) The prices of European call and put options on the same underlying asset, $C(\mathbf{S}_t, t)$ and $P(\mathbf{S}_t, t)$, are

$$C(\mathbf{S}_t, t) = \mathbf{S}_t \Phi(d_1) - E\mathrm{e}^{-r(T-t)}\Phi(d_2)$$

and

$$P(\mathbf{S}_t, t) = -\mathbf{S}_t \Phi(-d_1) + E\mathrm{e}^{-r(T-t)}\Phi(-d_2) \ ,$$

where

$$d_1 = \frac{\ln(\mathbf{S}_t/E) + (r + \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \quad \text{and} \quad d_2 = \frac{\ln(\mathbf{S}_t/E) + (r - \frac{1}{2}\sigma^2)(T-t)}{\sigma\sqrt{T-t}} \ .$$

i. Using the properties of $\Phi(x)$, show that these prices obey put-call parity.

ii. Find simpler formulas for $C(\mathbf{S}_t, t)$ and $P(\mathbf{S}_t, t)$, valid as $\mathbf{S}_t/E \to \infty$.

(b) Explain why the price of an American put option is not the same as that of a European put option.

3. Which of the following is the fair price of a financial derivative?

(a) $p_1(\mathbf{S}_t, t) = \mathbf{S}_t^{-r/2\sigma^2}$.

(b) $p_2(\mathbf{S}_t, t) = \mathbf{S}_t^{-r/\sigma^2}$.

(c) $p_3(\mathbf{S}_t, t) = \mathbf{S}_t^{-2r/\sigma^2}$.

(d) $p_4(\mathbf{S}_t, t) = \mathbf{S}_t$.

(e) $p_5(\mathbf{S}_t, t) = \mathbf{S}_t^2$.

(f) $p_6(\mathbf{S}_t, t) = \mathrm{e}^{rt}$.

(g) $p_7(\mathbf{S}_t, t) = \mathbf{S}_t\mathrm{e}^{\sigma t}$.

4. Let $\mathbf{V}_t = \left(t + \mathbf{W}_t{}^2\right)^{\frac{1}{2}}$, where $\mathbf{W}$ is the Wiener process and $t > 0$.

(a) Write down the integral that gives $\mathcal{P}[\mathbf{V}_t < x]$ for any $x > 0$.

(b) Find the probability density of $\mathbf{V}_t$.

(c) Find the stochastic differential equation for $\mathbf{V}_t$.

(d) Sketch some sample trajectories of $\mathbf{V}_t$.

5. A random variable $\mathbf{X}$ has the probability density $f(x) = N\mathrm{e}^{-\lambda x}$, where N and λ are constants and $x \geq 0$.

(a) Using the general properties of probability densities, show that $N = \lambda$.

(b) Find $\mathbb{E}(\mathbf{X})$.

(c) Find the standard deviation of $\mathbf{X}$.

6. (a) Using the Black-Scholes asset price model, show that the probability that a European call option, with strike price E at expiry date T on an asset whose price at time $t < T$ is $\mathbf{S}_t$, is exercised is equal to the probability that

$$\mathbf{G} > -d,$$

where $\mathbf{G}$ is a Gaussian random variable with mean zero and variance 1, and

$$d = \frac{\ln(\mathbf{S}_t/E) + (\mu - \frac{1}{2}\sigma^2)(T - t)}{\sigma\sqrt{T - t}}.$$

(b) In general one may wish to define the rate of return of an asset $\mathbf{S}$ as η, where the asset price at time t is

$$\mathbf{S}_t = \mathbf{S}_0\mathrm{e}^{\eta t}.$$

Show that, according to the Black-Scholes asset price model, the rate of return is a Gaussian random variable. Find its mean and standard deviation.

(c) Using the definition of $\Phi(x)$ on the formula sheet, show that $\Phi(-x) = 1 - \Phi(x)$.

7. A financial institution offers a contract called a "cash-or-nothing" option, that gives a payoff of a fixed amount A at time T if $\mathbf{S}_T > E$, and zero if $\mathbf{S}_T \leq E$.

(a) Use risk-neutral valuation to calculate the fair price for the contract. [*Hint: you may use the result given in question 6(a).*]

(b) Check that your answer satisfies the Black-Scholes partial differential equation.

END OF EXAM

Formulas

(1) Put-call parity:

$$\mathbf{C}_t + E\mathrm{e}^{-r(T-t)} = \mathbf{P}_t + \mathbf{S}_t\,.$$

(2) If the function $f(x)$ is the probability density of the random variable $\mathbf{X}$ then

$$\mathcal{P}[a < \mathbf{X} < b] = \int_a^b f(x)\mathrm{d}x.$$

The mean of the random variable is

$$\mathbb{E}(\mathbf{X}) = \int_{-\infty}^{\infty} x f(x)\,\mathrm{d}x.$$

For a Gaussian random variable with mean μ and standard deviation σ,

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left(-(x-\mu)^2/2\sigma^2\right)\,.$$

If $\mathbf{Y} = h(\mathbf{X})$, then the mean of the random variable $\mathbf{Y}$ is

$$\mathbb{E}(\mathbf{Y}) = \int_{-\infty}^{\infty} h(x)f(x)\mathrm{d}x\,.$$

(3) If $\mathbf{G}$ is a Gaussian random variable with mean 0 and variance 1 then $\mathbb{E}(\exp(m + s\mathbf{G})) = \mathrm{e}^{m+\frac{1}{2}s^2}$.

(4) The function $\Phi(x)$ is

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp\left(-\frac{1}{2}y^2\right)\mathrm{d}y\,.$$

Two properties are

$$\Phi(-x) = 1 - \Phi(x)$$

and

$$\Phi'(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}x^2\right).$$

(5) The Ito change-of-variables formula. If

$$\mathrm{d}\mathbf{R}_t = a(\mathbf{R}_t, t)\mathrm{d}t + b(\mathbf{R}, t)\mathrm{d}\mathbf{W}_t$$

and

$$\mathbf{V}_t = h(\mathbf{R}_t, t),$$

then

$$\mathrm{d}\mathbf{V}_t = \left(\frac{\partial h}{\partial x}a + \frac{\partial h}{\partial t} + \frac{1}{2}\frac{\partial^2 h}{\partial x^2}b^2\right)\mathrm{d}t + \left(\frac{\partial h}{\partial x}b\right)\mathrm{d}\mathbf{W}_t\,.$$

(6) The Black-Scholes asset price model:

$$\mathbf{S}_t = \mathbf{S}_0 \exp\left((\mu - \frac{1}{2}\sigma^2)t + \sigma\mathbf{W}_t\right)\,.$$

(7) The Black-Scholes partial differential equation:

$$\frac{\partial h}{\partial t} + \frac{1}{2}\sigma^2\mathbf{S}_t^2\frac{\partial^2 h}{\partial \mathbf{S}_t^2} + r\mathbf{S}_t\frac{\partial h}{\partial \mathbf{S}_t} - rh = 0.$$