MATH445001

This question paper consists of 7 printed pages, each of which is identified by the reference **MATH445001**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH4450
(June 2004)

Polymeric Fluids

Time allowed: **3 hours**

Answer FIVE of the SEVEN questions.

All questions carry equal marks.

1. **(a)** The shear stress, σ, in a Bingham plastic material is related to the shear-rate $\dot{\gamma}$ by

$$\mu\dot{\gamma} = \begin{cases} \sigma - \sigma_y, & \sigma > \sigma_y, \\ 0, & |\sigma| \le \sigma_y, \\ \sigma + \sigma_y, & \sigma < -\sigma_y, \end{cases}$$

where σ_y and μ are positive constants. Explain the physical significance of σ_y and sketch a graph of σ versus $\dot{\gamma}$.

(b) A cylindrical pipe of radius a is filled with fluid of density ρ and mounted with its axis vertical. Write down the equation for vertical momentum conservation on the assumption that inertia can be neglected and the extra stress, $\boldsymbol{\sigma}$, is a function of r only. If the top and bottom of the pipe are open to the atmosphere and z is measured downwards show the shear stress, σ_{rz}, in the pipe satisfies

$$\frac{1}{r}\frac{d}{dr}\left(r\sigma_{rz}\right) = -\rho g.$$

Find the shear stress, σ_{rz}.

(c) If the fluid in the pipe is a Bingham plastic, show that there will be no flow if $\sigma_y > \frac{1}{2}\rho g a$. Find the position of the yield surface when $\sigma_y < \frac{1}{2}\rho g a.$ Find the form of the fluid velocity, w for $\sigma_y = \frac{1}{4}\rho g a$ and sketch a graph showing $w(r)$. Calculate the volume flow, $Q = 2\pi\int_0^a rw\,dr$ down the pipe for this case.

2. **(a)** Using the appropriate formulae for cylindrical polar coordinates write down the velocity gradient for a flow in which the fluid velocity is given in cylindrical polar coordinates (r, θ, z) by $\mathbf{u} = (0, v(r), 0)$. Show that the $r\theta$ component of the strain-rate tensor, $\mathbf{E}$,

$$E_{r\theta} = \frac{1}{2}\dot{\gamma} = \frac{r}{2}\frac{d}{dr}\left(\frac{v}{r}\right),$$

where $\dot{\gamma}$ is the local shear-rate. Define the shear viscosity, $\mu(\dot{\gamma})$, and first and second normal stress differences, $N_1(\dot{\gamma})$ and $N_2(\dot{\gamma})$, in terms of the components of the extra stress tensor $\boldsymbol{\sigma}$.

(b) Write down the components of the momentum equation for such a flow on the assumption that fluid inertia is negligible and the gravitational acceleration, $\mathbf{g} = (0, 0, -g)$. (You may assume that $\sigma_{rz} = \sigma_{\theta z} = 0$). Show that this leads to the following equations

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\mu(\dot{\gamma})\dot{\gamma}\right) = 0,$$
$$\frac{\partial}{\partial r}\left(-p + \sigma_{zz} + N_2\right) = \frac{N_1}{r},$$
$$\frac{\partial}{\partial z}\left(-p + \sigma_{zz}\right) = \rho g.$$

(c) A vertical rod of radius a rotates at an angular velocity $\Omega > 0$ in a polymeric fluid in which

$$\mu(\dot{\gamma}) = \mu_0, \qquad N_1(\dot{\gamma}) = A|\dot{\gamma}|, \qquad N_2(\dot{\gamma}) = 0,$$

where μ_0 and A are both positive constants. Find the fluid velocity $v(r)$ and show that

$$\dot{\gamma} = -\frac{2\Omega a^2}{r^2}.$$

If the top surface is open to the atmosphere, show that the position of this surface is given by

$$h(r) = h_\infty + \frac{A\Omega a^2}{\rho g r^2},$$

where h_∞ is the height for $r \to \infty$.

3. **(a)** Write down the expression for the extra stress in a linear viscoelastic fluid of relaxation modulus $G(t)$. Explain how $G(t)$ can be measured directly from a step-strain experiment.

(b) The extra stress $\boldsymbol{\sigma}$ in the linear Maxwell model is related to the strain-rate by

$$\tau\frac{\partial\boldsymbol{\sigma}}{\partial t} + \boldsymbol{\sigma} = 2\mu\mathbf{E}(t).$$

Calculate the relaxation modulus $G(t)$ and show that

$$\int_0^\infty G(t)dt = \mu.$$

Sketch a graph of $G(t)$ and hence explain the significance of the parameter τ.

(c) Find the shear stress $\sigma_{xy}(t)$ generated by the fluid velocity $\mathbf{u} = (\dot{\gamma}y, 0, 0)$ in the following cases:

(i) $\dot{\gamma} = \begin{cases} 0 & t < 0, \\ \dot{\gamma}_0 & t \geq 0. \end{cases}$

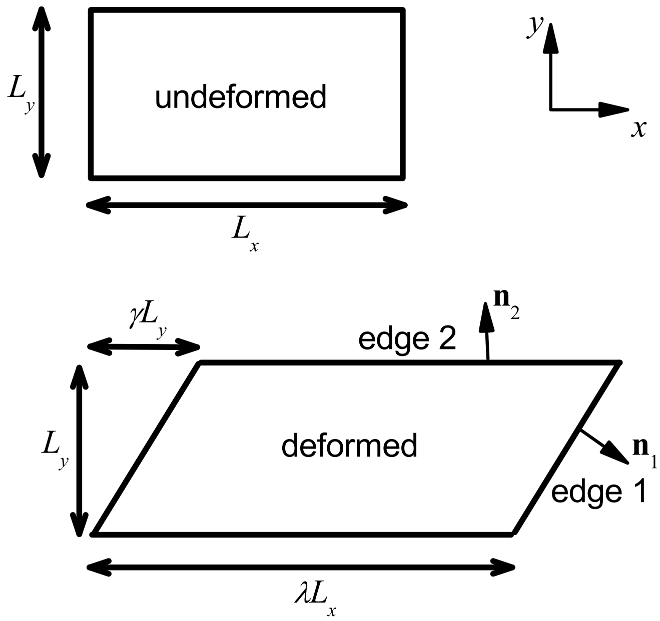
(ii) $\dot{\gamma} = \begin{cases} 0 & t < 0, \\ \frac{\gamma_0}{T} & 0 \leq t \leq T, \\ 0 & t > T. \end{cases}$

For each case sketch graphs of σ_{xy} and $\dot{\gamma}$ as functions of time. In case (ii) show that for $T \ll \tau$ the shear stress is approximately equal to $\gamma_0 G(t)$.

4. The expression for the total stress in a rubber is

$$\boldsymbol{\tau} = G\mathbf{F} \cdot \mathbf{F}^T - \beta \mathbf{I}.$$

A rectangular piece of rubber, of initial length L_x, width L_y and thickness L_z is deformed by means of clamps holding the edges, so that its new length is λL_x, its width remains L_y but it is subjected to a shear strain γ as shown in the diagram below.



(a) Assuming the volume of the rubber is conserved, what is the new thickness, L_z' of the rubber in the deformed shape?

(b) By considering the effect of the deformation on unit vectors in the x, y and z directions (or otherwise), determine the deformation gradient tensor, $\mathbf{F}$, for this deformation.

(c) Obtain the total stress, $\boldsymbol{\tau}$, in the rubber, and use the condition $\tau_{zz} = 0$ to determine the value of β.

(d) The total force on an area A with unit normal $\mathbf{n}$ is $\mathbf{f} = A\boldsymbol{\tau}\cdot\mathbf{n}$. Show that the edge labeled "1" in the diagram has area A_1 and unit normal $\mathbf{n}_1$ given by

$$\begin{aligned} A_1 &= \frac{\sqrt{1+\gamma^2}}{\lambda} L_y L_z \\ \mathbf{n}_1 &= \frac{1}{\sqrt{1+\gamma^2}} \begin{pmatrix} 1 \\ -\gamma \\ 0 \end{pmatrix} \end{aligned}$$

and hence determine the vector force required to clamp the rubber along edge 1. Similarly, determine the vector force required to clamp the rubber along the edge labeled "2".

5. The Langevin equation for a particle moving with friction constant ζ in a quadratic potential $U = \frac{1}{2}kx^2$ is

$$\zeta \frac{dx}{dt} = -kx + f(t)$$

where $\langle f(t) f(t') \rangle = 2k_B T \zeta \delta(t - t')$ where $k_B T$ is the thermal energy.

(a) Show that the solution of this equation, subject to initial condition $x(0) = 0$, is

$$x(t) = \frac{1}{\zeta} \int_0^t dt' f(t') \exp\left(\frac{t'-t}{\tau}\right)$$

where $\tau = \frac{\zeta}{k}$.

(b) Hence show that

$$\left\langle x(t)^2 \right\rangle = \frac{k_B T}{k} \left(1 - \exp\left(-\frac{2t}{\tau}\right)\right)$$

(c) **(i)** Obtain the limiting form of $\left\langle x(t)^2 \right\rangle$ for $t \ll \tau$ and compare your result with free-particle diffusion (where $\left\langle x(t)^2 \right\rangle = 2Dt$).

(ii) Obtain the limiting form of $\left\langle x(t)^2 \right\rangle$ for $t \gg \tau$. In this limit, show that the average energy $\langle U \rangle$ approaches $\frac{1}{2}k_B T$.

6. The Zimm model for polymer solutions has a time-dependent modulus of form

$$G(t) = G_0 \sum_{p=1}^{\infty} \exp\left(-\frac{p^{\frac{3}{2}} t}{\tau_z}\right)$$

(a) Obtain the viscosity, $\mu = \int_0^\infty G(s)\, ds$, and recoil after steady shear $R = \dot{\gamma} \int_0^\infty sG(s)\, ds / \int_0^\infty G(s)\, ds$.

(b) Use $G^* = G' + iG'' = \int_0^\infty i\omega G(s) \exp(-i\omega s)\, ds$ to show that

$$\begin{aligned} G' &= G_0 \sum_{p=1}^{\infty} \frac{\omega^2 \tau_z^2}{p^3 + \omega^2 \tau_z^2} \\ G'' &= G_0 \sum_{p=1}^{\infty} \frac{p^{\frac{3}{2}} \omega \tau_z}{p^3 + \omega^2 \tau_z^2}. \end{aligned}$$

QUESTION 6 CONTINUED...

Show that $G'' = \mu\omega$ for $\omega \ll \tau_z^{-1}$. By approximating the sum as an integral, find an approximation of form $G'' = c\omega^\alpha$ for $\omega \gg \tau_z^{-1}$. Hence sketch a graph of $\log G''$ versus $\log \omega$.

You may use the results $\sum_1^\infty p^{-3} \approx 1.202$, $\sum_1^\infty p^{-\frac{3}{2}} \approx 2.612$ and $\int_0^\infty \frac{x^{\frac{3}{2}}}{1+x^3} dx = \frac{2\pi}{3}$.

7. The constitutive equation for the Upper Convected Maxwell model is

$$\boldsymbol{\tau} = -\beta \mathbf{I} + G\mathbf{A},$$

where the structure tensor $\mathbf{A}$ satisfies

$$\frac{d\mathbf{A}}{dt} = \mathbf{K}\cdot\mathbf{A} + \mathbf{A}\cdot\mathbf{K}^T - \frac{1}{\tau}\left(\mathbf{A} - \mathbf{I}\right),$$

and $\mathbf{K}$ is the velocity gradient tensor.

(a) A cylinder of length L_0 and radius a_0 is formed from a polymeric material that obeys the Upper Convected Maxwell model. Initially the polymers are unstretched so that $\mathbf{A} = \mathbf{I}$. For $t > 0$ an axial force, $F(t)$ is applied to stretch the cylinder uniformly, so that its length increase as $L(t) = L_0 \exp(Et)$. Show, from conservation of mass, that the radius must decrease as $a(t) = a_0 \exp(-\frac{1}{2}Et)$ and the velocity gradient expressed in cylindrical polar coordinates (r, θ, z) is given by

$$\mathbf{K} = \begin{pmatrix} -\frac{1}{2}E & 0 & 0 \\ 0 & -\frac{1}{2}E & 0 \\ 0 & 0 & E \end{pmatrix}.$$

(b) Write down the equations for evolution of the tensor components, $A_{rr}, A_{\theta\theta}$ and A_{zz}. Show for $E > \frac{1}{2\tau}$ that these are given by

$$\begin{aligned} A_{rr} = A_{\theta\theta} &= \frac{1}{1+E\tau}\left(1 + E\tau \exp\left[-\left(\frac{1}{\tau} + E\right)t\right]\right), \\ A_{zz} &= \frac{1}{2E\tau - 1}\left(2E\tau \exp\left[\left(2E - \frac{1}{\tau}\right)t\right] - 1\right). \end{aligned}$$

(c) Show that the magnitude of the axial force required to stretch the cylinder is

$$F(t) = G\pi a^2 (A_{zz} - A_{rr}),$$

and hence calculate $F(t)$. By examining the limit $t \to \infty$, show that the $F(t)$ decays to zero for $E < \frac{1}{\tau}$, but increases exponentially for $E > \frac{1}{\tau}$. What happens for $E = \frac{1}{\tau}$?

Formula Sheet

Cartesian coordinates

pressure, p, velocity, $\mathbf{u} = u\mathbf{e}_x + v\mathbf{e}_y + w\mathbf{e}_z$, velocity gradient, $\mathbf{K}$ with $K_{ij} = \frac{\partial u_i}{\partial x_j}$

$$\nabla p = \frac{\partial p}{\partial x}\mathbf{e}_x + \frac{\partial p}{\partial y}\mathbf{e}_y + \frac{\partial p}{\partial z}\mathbf{e}_z, \qquad \nabla \cdot \mathbf{u} = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z},$$

$$\mathbf{K} = \begin{pmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} & \dfrac{\partial u}{\partial z} \\[2ex] \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} & \dfrac{\partial v}{\partial z} \\[2ex] \dfrac{\partial w}{\partial x} & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial z} \end{pmatrix} \qquad \nabla \cdot \boldsymbol{\sigma} = \begin{pmatrix} \dfrac{\partial \sigma_{xx}}{\partial x} + \dfrac{\partial \sigma_{yx}}{\partial y} + \dfrac{\partial \sigma_{zx}}{\partial z} \\[2ex] \dfrac{\partial \sigma_{xy}}{\partial x} + \dfrac{\partial \sigma_{yy}}{\partial y} + \dfrac{\partial \sigma_{zy}}{\partial z} \\[2ex] \dfrac{\partial \sigma_{xz}}{\partial x} + \dfrac{\partial \sigma_{yz}}{\partial y} + \dfrac{\partial \sigma_{zz}}{\partial z} \end{pmatrix}$$

Cylindrical Polar Coordinates

velocity, $\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_z$.

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{\partial p}{\partial z}\mathbf{e}_z, \qquad \nabla \cdot \mathbf{u} = \frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{\partial w}{\partial z},$$

$$\mathbf{K} = \begin{pmatrix} \dfrac{\partial u}{\partial r} & \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r} & \dfrac{\partial u}{\partial z} \\[2ex] \dfrac{\partial v}{\partial r} & \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} + \dfrac{u}{r} & \dfrac{\partial v}{\partial z} \\[2ex] \dfrac{\partial w}{\partial r} & \dfrac{1}{r}\dfrac{\partial w}{\partial \theta} & \dfrac{\partial w}{\partial z} \end{pmatrix}$$

$$\nabla \cdot \boldsymbol{\sigma} = \begin{pmatrix} \dfrac{1}{r}\dfrac{\partial}{\partial r}(r\sigma_{rr}) + \dfrac{1}{r}\dfrac{\partial \sigma_{\theta r}}{\partial \theta} + \dfrac{\partial \sigma_{zr}}{\partial z} - \dfrac{\sigma_{\theta\theta}}{r} \\[2ex] \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\sigma_{r\theta}\right) + \dfrac{1}{r}\dfrac{\partial \sigma_{\theta\theta}}{\partial \theta} + \dfrac{\partial \sigma_{z\theta}}{\partial z} + \dfrac{\sigma_{\theta r} - \sigma_{r\theta}}{r} \\[2ex] \dfrac{1}{r}\dfrac{\partial}{\partial r}(r\sigma_{rz}) + \dfrac{1}{r}\dfrac{\partial \sigma_{\theta z}}{\partial \theta} + \dfrac{\partial \sigma_{zz}}{\partial z} \end{pmatrix}$$

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Spherical Polar Coordinates

$\mathbf{u} = u\mathbf{e}_r + v\mathbf{e}_\theta + w\mathbf{e}_\phi$

$$\nabla p = \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}\mathbf{e}_\phi,$$

$$\nabla \cdot \mathbf{u} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(v\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial w}{\partial \phi},$$

$$\mathbf{K} = \begin{pmatrix} \dfrac{\partial u}{\partial r} & \dfrac{1}{r}\dfrac{\partial u}{\partial \theta} - \dfrac{v}{r} & \dfrac{1}{r\sin\theta}\dfrac{\partial u}{\partial \phi} - \dfrac{w}{r} \\[2ex] \dfrac{\partial v}{\partial r} & \dfrac{1}{r}\dfrac{\partial v}{\partial \theta} + \dfrac{u}{r} & \dfrac{1}{r\sin\theta}\dfrac{\partial v}{\partial \phi} - \dfrac{w}{r}\cot\theta \\[2ex] \dfrac{\partial w}{\partial r} & \dfrac{1}{r}\dfrac{\partial w}{\partial \theta} & \dfrac{1}{r\sin\theta}\dfrac{\partial w}{\partial \phi} + \dfrac{u}{r} + \dfrac{v}{r}\cot\theta \end{pmatrix}$$

$$\nabla \cdot \boldsymbol{\sigma} = \begin{pmatrix} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\sigma_{rr}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sigma_{\theta r}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi r}}{\partial \phi} - \frac{\sigma_{\theta\theta}+\sigma_{\phi\phi}}{r} \\[2ex] -\frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\sigma_{r\theta}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sigma_{\theta\theta}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\theta}}{\partial \phi} + \frac{\sigma_{\theta r}-\sigma_{r\theta}-\sigma_{\phi\phi}\cot\theta}{r} \\[2ex] \frac{1}{r^3}\frac{\partial}{\partial r}\left(r^3\sigma_{r\phi}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sigma_{\theta\phi}\sin\theta\right) + \frac{1}{r\sin\theta}\frac{\partial \sigma_{\phi\phi}}{\partial \phi} + \frac{\sigma_{\phi r}-\sigma_{r\phi}+\sigma_{\phi\theta}\cot\theta}{r} \end{pmatrix}$$

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