MATH–443001

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–443001

Only approved basic scientific calculators may be used.

Examination for the Module MATH–4430

(May/June 2004)

Advanced Dynamical Systems

Time allowed: 2 hours

Answer **three** questions.
All questions carry equal marks.

1. The following set of ordinary differential equations (ODEs) has been proposed as a model of a coupled disk dynamo:

$$\begin{aligned} \dot{x} &= (\beta - 1)x - \alpha y - xz, \\ \dot{y} &= x - y, \\ \dot{z} &= -z + \beta x^2, \end{aligned}$$

where $\alpha > 0$ and $\beta > 0$ are parameters, and $x(t)$, $y(t)$ and $z(t)$ represent aspects of the physical model.

(a) Find the equilibrium states of the ODEs.

(b) Determine the stability of the equilibria, giving the locations of the codimension-one and codimension-two bifurcations, and showing that there are Hopf bifurcations when

$$\beta = 2 \quad \text{with} \quad \alpha > 1, \qquad \text{and} \qquad \alpha = 1 \quad \text{with} \quad \beta > 2.$$

(c) Draw the bifurcation lines in the (α, β) parameter plane, indicating where the equilibria are stable or unstable.

(d) Argue that there must be additional (global) bifurcations, and indicate where these might be located in the (α, β) parameter plane

continued . . .

2. Consider the following third-order set of ODEs:

$$\begin{aligned}\dot{u} &= \kappa u - \lambda v - vw,\\ \dot{v} &= u,\\ \dot{w} &= -w + v^2,\end{aligned}$$

where κ and λ are parameters.

(a) Find the equilibrium states of the ODEs.

(b) Show that there is a codimension-two bifurcation at the parameter values $\kappa = \lambda = 0$.

(c) By writing $w = h(u, v)$, perform a centre manifold reduction at $\kappa = \lambda = 0$, and show that the dynamics at the codimension-two point is governed by ODEs of the form:

$$\begin{aligned}\dot{u} &= Su^3 + Ru^2v + Quv^2 + Pv^3,\\ \dot{v} &= u,\end{aligned}$$

where P, Q, R and S are constants to be determined.

(d) At $\kappa = \lambda = 0$, perform a near-identity change of coordinates of the form

$$\begin{aligned}x &= u + \alpha_1 u^3 + \beta_1 u^2 v + \gamma_1 uv^2 + \delta_1 v^3,\\ y &= v + \alpha_2 u^3 + \beta_2 u^2 v + \gamma_2 uv^2 + \delta_2 v^3\end{aligned}$$

to transform the equations into the form

$$\begin{aligned}\dot{x} &= Qxy^2 + Py^3,\\ \dot{y} &= x.\end{aligned}$$

3. Discuss the Shil'nikov global bifurcation, taking as an example the set of ODEs:

$$\begin{aligned}\dot{x} &= \lambda_- x - \omega y + f_1(x, y, z; \mu)\\ \dot{y} &= \omega x + \lambda_- y + f_2(x, y, z; \mu)\\ \dot{z} &= \lambda_+ z \qquad\qquad + f_3(x, y, z; \mu)\end{aligned}$$

where μ is a parameter, $\lambda_- < 0 < \lambda_+$, $\omega > 0$ and f_i are purely nonlinear functions of x, y and z. Assume that when $\mu = 0$, there is a homoclinic orbit that leaves the origin with $z > 0$ and returns to the origin tangent to the (x, y) plane, and that when $\mu < 0$, the unstable manifold of the origin returns above the (x, y) plane.

Include in your discussion an outline of the derivation of an approximate return map from a surface of section Σ (close to the origin) to itself. Indicate how this map describes the periodic orbits of the ODEs for small $|\mu|$, distinguishing the cases $|\lambda_-| > |\lambda_+|$ and $|\lambda_-| < |\lambda_+|$. Show that in one of these two cases, for $|\mu|$ sufficiently small, typical systems of this sort have periodic orbits of period T if

$$\mu \approx Ae^{\lambda_- T}\cos(\omega T - \Phi)$$

where A and Φ are constants.

continued ...

4. Consider a continuous one-dimensional map:

$$x_{n+1} = f(x_n)$$

from an interval I to itself.

(a) State carefully what it means for such a map to have a *horseshoe*. Give an example of a map that has a horseshoe.

(b) Define the *topological entropy* of such a map. Show that if a map has a horseshoe, it has positive topological entropy, defining carefully any terms you need.

Consider the family of tent maps:

$$x_{n+1} = T_s(x_n) = \begin{cases} sx_n & \text{if } 0 \leq x_n \leq \frac{1}{2} \\ s(1 - x_n) & \text{if } \frac{1}{2} \leq x_n \leq 1 \end{cases}$$

(c) Show that T_s is a map from $I = [0, 1]$ to itself provided that $0 \leq s \leq 2$.

(d) Show that if $\sqrt{2} < s \leq 2$, the map T_s^2 has a horseshoe.

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