**MATH443001**

This question paper consists of 2 printed pages, each of which is identified by the reference **MATH4430**.

Only approved basic scientific calculators may be used.



Examination for the Module MATH4430

(January 2003)

**Advanced Dynamical Systems**

Time allowed: **2 hours**

Answers should be submitted to not more than **three** questions.
All questions carry equal marks.

**1.** Consider the third-order set of ordinary differential equations (ODEs):

$$\begin{aligned}\dot{x} &= (1+\kappa)x - (1+\kappa+\lambda)y - xz,\\ \dot{y} &= x - y,\\ \dot{z} &= -z + (2+\kappa)x^2,\end{aligned}$$

where $\kappa$ and $\lambda$ are parameters, and $\kappa > -2$.

(i) Find the equilibrium points of the ODEs.

(ii) Determine the stability of the equilibria, giving the locations and types of the codimension-one and codimension-two bifurcations, and showing that there is a Hopf bifurcation when

$$\kappa = -\lambda \qquad \text{with} \qquad \lambda < 0.$$

(iii) Draw the bifurcation lines in the $(\kappa, \lambda)$ parameter plane, indicating where the equilibria are stable or unstable.

(iv) Explain why there must be additional (global) bifurcations, and indicate where these might be located in the $(\kappa, \lambda)$ parameter plane.

**2.** Suppose that, when a parameter $\mu = 0$, a second-order ODE $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, \mu)$ (with $\mathbf{x} \in \mathbb{R}^2$) has a saddle equilibrium point $\mathbf{x} = 0$, and that there is a homoclinic orbit connecting the equilibrium point $\mathbf{x} = 0$ to itself. Explain why this situation is structurally unstable. By deriving an appropriate one-dimensional map, explain what you would expect to find as $\mu$ is varied away from 0, stating clearly any necessary assumptions.

**3.** Suppose $x_{n+1} = f(x_n)$ (with $x \in \mathbb{R}$), where $f$ is a continuous map from an interval $I$ to itself, and suppose also that $f$ has an orbit of least period 3. Show that

(i) $f$ has orbits of least period $m$ for all positive integers $m$.

(ii) The topological entropy $h(f) \geq \log\left(\frac{1+\sqrt{5}}{2}\right)$.

**4.** Consider the third-order set of ODEs:

$$\begin{aligned} \dot{u} &= \kappa u - \lambda v + Muw + Nvw, \\ \dot{v} &= u, \\ \dot{w} &= -w + v^2, \end{aligned}$$

where $\kappa$ and $\lambda$ are parameters, and $M$ and $N$ are constants.

(i) Identify the codimension-two bifurcation at $(\kappa, \lambda) = (0, 0)$.

(ii) Perform a centre manifold reduction at this bifurcation point, and write down equations governing the evolution on the center manifold.

(iii) By performing suitable near-identity coordinate transformations, show that the differential equation on the centre manifold is equivalent to

$$\begin{aligned} \dot{x} &= Py^3 + Qxy^2, \\ \dot{y} &= x, \end{aligned}$$

where the values of $P$ and $Q$ are to be given explicitly.

END