MATH373301

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH3733**.

Only approved basic scientific calculators may be used.

UNIVERSITY OF LEEDS

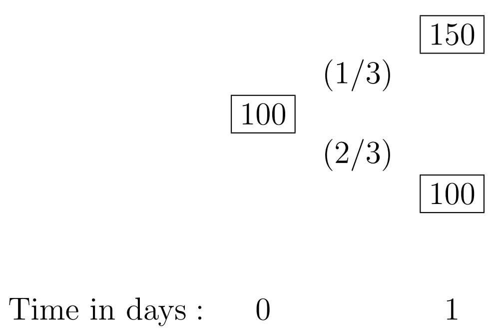
Examination for the Module MATH3733
(January 2006)

Stochastic Financial Modelling

Time allowed: **3 hours**

Attempt not more than **four** questions.
All questions carry equal marks.

1. (a) Explain how to interpret the meaning of the following one-step binomial tree model with interest rate $r = \ln(1.2)$ where stock values are in boxes and transition probabilities are in parentheses,



Define what is called a European call option with strike price \$120 and expiry $T = 1$, the contract being written at time 0.
Denote by C_T the price that this option will have at expiry. Compute C_T in terms of the price of the asset at expiry, S_T.
Calculate the price C_0 using the equivalent portfolio principle.

(b) Define the notion of implied probabilities for the model in (a).
Why they are useful?
Show that they equal 2/5 for the upper branch, and 3/5 for the lower one.
Compute again the price C_0, this time using the implied probabilities.

(c) Suppose in case (a) above, you see on the market a call option at time zero at the price $C_0^* = \$40$. Show that an arbitrage possibility arises, and construct a portfolio and an algorithm which realize this possibility.

2. (a) Write down Itô's formula for the process $f(t, W_t)$ with $f(t, x) = t^3 e^x$, where W_t is a standard Wiener process.

(b) Give the definition of a martingale.
It is known that under the Black-Merton-Scholes model, a European call option price $(C_t,\ t \geq 0)$ is a martingale under the implied probability measure, if the interest rate $r = 0$. Prove that in this case the price of the European call option with expiry $T = 1$ and strike price $K = 2$ can be expressed by the formula,

$$C_0 = \tilde{E}(S_1 - 2)_+,$$

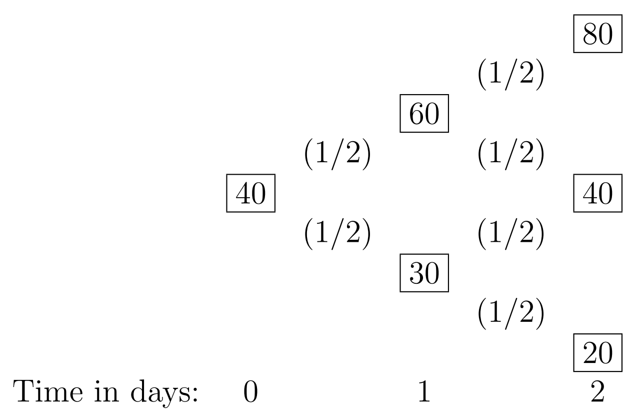
where $\tilde{E}$ denotes the expected value computed under implied probability, and $S_t, t \geq 0$, is a stock price.

(c) Using the formula from (b) above, or otherwise, prove that for the price C_0 from (b) equals the following Gaussian integral,

$$\int_{\ln 2 + 1/2}^{\infty} (e^{x - \frac{1}{2}} - 2)\, \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx.$$

Calculate this integral in terms of the Laplace function $\Phi(z) := \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}\, dx$, $-\infty < z < +\infty$.

3. (a) Explain how to interpret the following two-step binomial tree model with interest rate $r = 0$, where stock values are in boxes and transition probabilities are in parentheses,



Compute the implied probabilities for this model.

(b) Define what is called a European put option with strike price \$50 and expiry $T = 2$, the contract being written at time 0.
Denote by P_T the price that this option will have at expiry. Calculate the price at zero P_0 using the implied probabilities, or otherwise.

(c) Suppose in the setting (a)–(b), an insider knows at time zero that the stock price tomorrow will definitely rise. Show that an arbitrage possibility arises at time zero, and construct a portfolio and an algorithm which realize this possibility.

4. (a) Let $(W_t,\ t \geq 0)$ be a Wiener process on an appropriate path space with probability measure P, and define a process $\gamma_t = \exp(-W_t - \frac{t}{2})$, $t \geq 0$. Using either integration for Gaussian integrals, or stochastic differentials, show that

$$E(\gamma_t) = 1, \quad t > 0.$$

(b) Write down Itô's formula for the process $(W_t + t)\gamma_t$.
Hence or otherwise, show that $(W_t + t)\gamma_t$ is a martingale.

(c) Using the stochastic differential of the process $Z_t = e^{-2W_t+t}$, or otherwise, show that

$$E(Z_t) > 1 \quad \text{for any } t > 0.$$

5. (a) Find a solution of the equation,

$$dX_t = \frac{1}{2} X_t dW_t - X_t \, dt, \quad X_0 = 3,$$

where $(W_t, \, t \geq 0)$ is a standard Wiener process.
Hint: the general form for the solution of a linear SDE is $X_t = Ae^{BW_t+Ct}$ *with suitable constants* A, B, C.

(b) Calculate the variance of the stochastic integral

$$Y_t = \int_0^t e^{W_s - s} \, dW_s.$$

Hint: you may either use the properties of stochastic integrals, or, otherwise, compute some Gaussian integral.

(c) Consider the Black-Merton-Scholes model of a stock price,

$$S_t = S_0 \exp\left(-\frac{1}{2} t + \frac{1}{2} W_t\right), \quad 0 \leq t \leq 1,$$

with drift $\mu = -1/2$, and volatility $\sigma = \frac{1}{2}$, where $(W_t, \, 0 \leq t \leq 1)$ is a standard Wiener process. Consider a European call option with expiry $T = 1$ and strike price $K = 1$. Assume interest rate $r = 1$.
Write down the partial differential equation for the price of this call option considered as a function of time t and stock price S at time t.
Hence, or otherwise, explain why the European call option price does not depend on the drift μ.
With the help of an SDE representation or otherwise, compute this price at $t = 0$, leaving the answer in terms of a Gaussian integral, $c \int\limits_{-7/4}^{\infty} (e^{\frac{1}{2}x + \frac{7}{8}} - 1) \frac{1}{\sqrt{2\pi}} e^{-x^2/2} \, dx$.
Compute the value c.
Hint: find the generator of a diffusion under implied probability, solve the corresponding SDE via the Wiener process, and, hence, get the expression for the price via an SDE representation.

END