MATH373301

This question paper consists of 3 printed pages, each of which is identified by the reference **MATH3733**.

Only approved basic scientific calculators may be used.

UNIVERSITY OF LEEDS

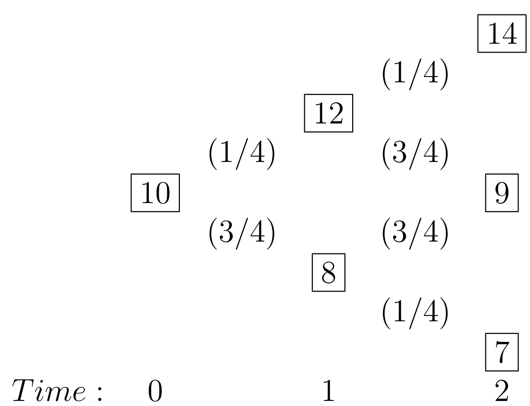
Examination for the Module MATH3733
(January 2005)

Stochastic Financial Modelling

Time allowed: **3 hours**

Do not answer more than four questions.
All questions carry equal marks.

1. (a) Explain how to interpret the meaning of the following two-step binomial tree model with interest rate $r = 0$ where stock values are in boxes and transition probabilities are in parentheses,



Define and calculate the implied probabilities for each branch of this model.

(b) Define a European call option with strike price \$10 and expiry $T = 2$, the contract being written at time 0.
Denote by C_T the price that this option will have at expiry. Calculate the expected value EC_T.

(c) Explain the principles of *non-arbitrage* and *equivalent portfolio.*
Explain how they are used in pricing options.
Compute the price C_0 of the call option described in (b) at time 0 either using these two principles, or via implied probabilities, or otherwise.
Compare the two values, C_0 and EC_T, – the fair price and the expected payoff, – and comment on any difference.

2. (a) Define the standard $\mathcal{N}(0,1)$ and general $\mathcal{N}(a,\sigma^2)$ Gaussian random variables via their densities (assume $\sigma^2 > 0$).
In both cases write down their expected values, variances, and characteristic functions.

(b) For the standard Gaussian random variable *prove* that the variance equals 1, using integration by parts or otherwise, assuming that $\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx = 1$.

(c) The Black-Scholes formula for pricing a European call option with strike price $K = 1$ and expiry $T = 1$ on the market with interest rate $r = 3$ and volatility $\sigma = 1$ can be expressed by the following Gaussian integral,

$$e^{-3} \int_{-5/2}^{\infty} (e^{x+\frac{5}{2}} - 1) \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx.$$

Calculate this integral in terms of the Laplace function $\Phi(z) := \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx$.

3. (a) Give the definition of a Wiener process, $(W_t, t \geq 0)$.
Explain the relation between a Random Walk and a Wiener process on an appropriate path space, using the Central Limit Theorem or otherwise.
Formulate the Central Limit Theorem.

(b) Consider a Wiener process $(W_t,\, t \geq 0)$ with its filtration $(\mathcal{F}_t^X,\, t \geq 0)$. On the same probability space consider another random process $(f_t,\, t \geq 0)$. State the assumptions on this process required in order to define a stochastic integral $X_t = \int_0^t f_s\, dW_s$.
(A) Formulate a representation for the variance of this stochastic integral using the Riemann (non-stochastic) integral.
(B) What is the mean value of X?
If $f_t = f$ does not depend on t, verify both statements (A) and (B) above.

(c) The Black-Scholes formula for the price at time t, $0 \leq t < 1$, of the European call option with strike price K and expiry $T = 1$ in a market with interest rate r has the form,

$$C_t \equiv C_t(S) = S\Phi\left(\frac{\log\left(\frac{S}{K}\right) + \left(r + \frac{\sigma^2}{2}\right)(1-t)}{\sigma\sqrt{1-t}}\right)$$
$$-e^{-r(1-t)} K\Phi\left(\frac{\log\left(\frac{S}{K}\right) + \left(r - \frac{\sigma^2}{2}\right)(1-t)}{\sigma\sqrt{1-t}}\right),$$

where $\Phi(z)$ is the Laplace function,

$$\Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-x^2/2}\, dx.$$

Prove that for any value $S > 0$,

$$\lim_{t \to 1} C_t(S) = (S - K)_+.$$

Explain why this limiting behaviour is expected.

4. (a) Let $(X_t,\ t \geq 0)$ be a Markov process. Under what conditions is $(X_t,\ t \geq 0)$ a martingale?
Formulate the Cameron - Martin - Girsanov Theorem about a Wiener process and transformation of measure.
Using this theorem or otherwise, prove that the process $(W_t - 2t, 0 \leq t \leq 1)$, is a martingale under the new probability measure P^1.
Assume without proof that X is Markov under the new probability measure P^1.

(b) Let $(W_t,\ 0 \leq t \leq 1)$ be a Wiener process on an appropriate path space with probability measure P, and define a random variable $\gamma_1 = e^{(2W_1 - 2)}$. Show that

$$E^P(\gamma_1) = 1,$$

and hence or otherwise explain why γ_1 may be considered as a density for some new probability measure P^1 with respect to the original probability measure P.

(c) Calculate the stochastic differential of the process $Z_t = e^{(W_t + 2t)}$.
Using the result, or otherwise, show that

$$E^P(Z_t) > 1 \quad \text{for any } t > 0.$$

5. (a) Calculate the mean value and the variance of the stochastic integral

$$Y_t = \int_0^t e^{W_s + (3s/2)}\, dW_s.$$

Hint: for the variance, you may use martingales, or compute some Gaussian integrals.

(b) Formulate what is called a generator of a process satisfying the following linear stochastic differential equation (SDE),

$$dX_t = X_t dW_t + X_t\, dt, \quad X_0 = 1.$$

Find a solution of this equation.
Hint: the general form for the solution of a linear SDE is $X_t = Ae^{BW_t + Ct}$ with suitable constants A, B, C.

(c) Consider the Black-Merton-Scholes model of a stock price,

$$S_t = S_0 e^{(3/2)t + W_t}, \quad 0 \leq t \leq 1,$$

with drift $\mu = 3/2$ and volatility $\sigma = 1$, where $(W_t,\ 0 \leq t \leq 1)$ is a Wiener process. Consider the European call option with expiry $T = 1$ and strike price $K = 1$. Assume interest rate $r = 1$.
It can be shown that the price of such an option satisfies the following partial differential equation,

$$\frac{\partial u(t,x)}{\partial t} + \frac{x^2}{2}\frac{\partial^2 u(t,x)}{\partial x^2} + x\frac{\partial u(t,x)}{\partial x} - u(t,x) = 0, \;\; 0 \leq t \leq 1, \quad u(1,x) = (x-1)_+.$$

With the help of a SDE representation or otherwise, compute the value $u(0,1)$, leaving the answer in terms of a Gaussian integral, $c\int_{-1/2}^{\infty}(e^{x+\frac{1}{2}} - 1)\,\frac{1}{\sqrt{2\pi}}\,e^{-x^2/2}\,dx$.
Compute the value c.
Hint: find the generator and solve the corresponding SDE via the Wiener process, and, hence, get the expression for $u(0,1)$ via a SDE representation.