MATH373301

This question paper consists of 2 printed pages, each of which is identified by the reference **MATH3733**.

Only approved basic scientific calculators may be used.

UNIVERSITY OF LEEDS

Examination for the Module MATH3733
(January 2002)

Stochastic Financial Modelling

Time allowed: **3 hours**

Attempt not more than **four** questions. All questions carry equal marks.

1. (a) Define European call and put options, explaining the terms expiry time, strike price, and interest rate.

(b) Explain put and call parity using no-arbitrage arguments.

(c) Construct a replicating self-financing portfolio $\mathcal{P}_t = \phi_t S_t + \psi_t B_t$, $B_0 = 1$, which replicates the following option in the one-step binomial branch model with time-step size $\delta = 1$, and interest rate r such that $\exp(r) = 4/3$ (prices in dollars):

		25
stock:	10	
		5
time:	0	1

option's payoff at expiry: $\begin{cases} 0, & \text{if } S_T = 25, \\ 1, & \text{if } S_T = 5, \end{cases}$

with all the probabilities of up and down jumps equal to 1/2.

2. (a) Define a self-financing portfolio for the discrete time market model given by a binomial tree model. Define a self-financing portfolio for the Black - Merton - Scholes continuous time market model.

(b) For the binomial two-step model with the structure of nodes and prices

			350
		250	
stock:	150		200
		100	
			50
time:	0	1	2

and with all the probabilities of up and down jumps equal to 1/2, find the correct price of the European option with expiry $T = 2$ and payoff

$$C_T = \begin{cases} 0, & \text{if } S_T = 350, \\ 30, & \text{if } S_T = 200, \\ 60, & \text{if } S_T = 50, \end{cases}$$

assuming interest rate $r = 0$ and S_T being the stock price at expiry.

QUESTION 2 CONTINUED...

(c) For the same two-step binomial model and the same option, assume that the writer of this option knows for certain at time $t = 0$ that at time $t = 1$ the stock price will be exactly \$ 250 (insider information). How should he value the option at time $t = 0$? Why may this price differ from (b)?

3. (a) Define Brownian motion. Using Brownian motion, define the Black - Merton - Scholes market model with one stock, and explain what are called volatility and trend in this model.

(b) Find the stochastic differential of the process $\sqrt{1 + W_t^4}$ (W_t is a Brownian motion).

(c) Formulate the martingale price principle in the Black - Merton - Scholes market model with one stock, interest rate $r = 0$, drift $\mu = 7$ and volatility $\sigma = 1$. Find the Cameron - Martin - Girsanov transformation which makes the stock price a martingale on a time interval $[0, T]$. Consider a European call option in this model with a given expiry time T and strike price K. Show how its value can be expressed via Brownian motion in the form (S_0 being the initial price of the stock)

$$C_0 = E(S_0 \exp(W_T - T/2) - K)_+.$$

4. (a) Define a standard Gaussian distribution $\mathcal{N}(0, 1)$ by writing down its density. Write down its mean value, variance and characteristic function. Formulate the Central Limit Theorem for independent, identically distributed random variables, using any of the equivalent definitions of weak convergence.

(b) Solve the stochastic differential equation (SDE)

$$dX_t = X_t dW_t + X_t dt, \quad X_0 = x,$$

where W_t is a Brownian motion.

(c) Express the solution $u(0, x)$ of the Black - Scholes equation for a European option

$$u_t(t, x) + \frac{1}{2} x^2 u_{xx}(t, x) + x\, u_x(t, x) - u(t, x) = 0, \quad 0 \le t \le 1, \ x \in R^1, \quad u(1, x) = g(x),$$

via Brownian motion W_t, either using Itô's formula, or otherwise. Interpret the function $g(x)$ in terms of the option.

5. (a) Find a solution to the stochastic differential equation (SDE)

$$dX_t = -dW_t, \quad X_0 = x,$$

where W_t is a Brownian motion, and explain why this solution is unique.

(b) Using Itô's formula, or otherwise, show the inequality for any $t > 0$,

$$E\left(\exp(2W_t - t)\right) > 1.$$

(c) Find a solution to the partial differential equation

$$u_t + (1/2)\, u_{xx} = 0, \quad 0 \le t \le 1, \ x \in R, \quad u(1, x) \equiv x,$$

either by writing a corresponding SDE and applying Itô's formula, or otherwise.

END