MATH372301

This question paper consists of 4 printed pages, each of which is identified by the reference **MATH3723**.

Only approved basic scientific calculators may be used. Statistical table provided on p. 4.

UNIVERSITY OF LEEDS

Examination for the Module MATH3723
(May/June 2003)[1]

Statistical Theory

Time allowed: **3 hours**

Attempt not more than **four** questions. All questions carry equal marks.

1. (a) Define what is meant by an unbiased estimator of a parameter, a best unbiased estimator and a best linear unbiased estimator in the Mean Squared Error (MSE) sense.

Write down the density of a normal random variable $\mathcal{N}(\mu, \sigma^2)$.

Two independent random variables $X_1, X_2 \sim \mathcal{N}(\mu, \sigma^2)$ with $\sigma^2 = 1$ are given.

Define all linear unbiased estimators of μ based on X_1 and X_2.

Compute the best linear unbiased estimator of μ in the MSE sense.

(b) Define the Fisher information and show how to compute it for the model $\mathcal{N}(\mu, 1)$.

Determine the Cramér-Rao lower bound for this model.

Using this bound, or otherwise, find whether the best linear estimator from part (a) is also the best among all – not necessarily linear – unbiased estimators.

(c) Compute the Cramér-Rao lower bound for the parameter θ for the family of densities $\theta e^{-\theta x},\ x > 0,\ \theta > 0$.

Is this bound attained by any estimator? Justify your answer.

[You may assume the following result without proof: if an estimator $\hat{\theta}$ attains the Cramér-Rao lower bound, it must satisfy an equation, $\frac{\partial \ell}{\partial \theta} = I(\theta)(\hat{\theta} - \theta)$.]

2. (a) Define in detail the notion of Maximum Likelihood Estimator for a parametric statistical model, and state the corresponding differential equation.

Comment on this equation, that is, how it relates to the definition of the Maximum Likelihood Estimator.

(b) Compute the Maximum Likelihood Estimator for the sample $X_1, \ldots, X_n$ from the distribution with the density $e^{-|x-\theta|}/2,\ -\infty < x < \infty$.

Compute a second estimator of θ using the method of moments, and compare the asymptotic efficiency of both estimators.

[You may use without proof that the asymptotic Mean Square Error of the median M_n is equivalent to $(4f^2(M)n)^{-1}$, if the density f is continuous and positive at the point of median M. Also it is enough to consider n odd.]

QUESTION 2 CONTINUED...

(c) Find the Maximum Likelihood Estimator of θ based on a random sample from the uniform distribution $U[0, \theta]$, and compute its Mean Square Error.

Comment on the rate of convergence of the Mean Square Error.

3. (a) Define the notion of a sufficient statistics.

State the Factorization Criterion for sufficient statistics and the Rao-Blackwell Theorem on improvement of unbiased estimators.

(b) Compute the minimal sufficient statistic for a sample $X_1, \ldots, X_n$ from the Bernoulli distribution $B(p)$. [Neither an exact proof of minimality, nor the definition of minimality, is required.]

(c) In the model from (b), for the estimator X_1, show its unbiasedness, and use the Rao-Blackwell Theorem to improve it, given the minimal sufficient statistic based on two observables X_1, X_2.

Compare the Mean Square Error of the improved estimator with the Cramér-Rao lower bound and comment on it.

4. (a) Define in detail the notion of a Bayesian estimator based on the posterior Mean Squared Error criterion.

Prove that, in appropriate notation which you should define, the resulting estimator may be written $E(\theta|X)$.

(b) Consider a sample $X = (X_1, \ldots, X_n)$ from the Bernoulli distribution, $\theta^x(1-\theta)^{1-x}$, $x = 0, 1$, $0 \le \theta \le 1$. Let the prior for θ be $q(\theta = 1) = q(\theta = 0) = 1/2$. Compute the Bayesian estimator referred to in part (a) and determine its posterior Mean Square Error.

(c) For the model $X \sim \mathcal{N}(\theta, \sigma^2)$, with known σ^2, a conjugate prior is defined as $q(\theta) \sim \mathcal{N}(\phi, \tau^2)$. Write down an expression for the joint density (θ, X), and show that posterior distribution of θ is normal,

$$\mathcal{N}\left(\frac{\phi/\tau^2 + X/\sigma^2}{1/\tau^2 + 1/\sigma^2}, \left(\frac{1}{\tau^2} + \frac{1}{\sigma^2}\right)^{-1}\right).$$

Using this representation, write down the Bayesian Estimator for this model and its posterior Mean Square Error.

[Hint: it suffices to find the joint density of (θ, X), because the unconditional density of X does not involve θ; hence, or otherwise, show that the posterior distribution of θ given X is normal with the desired parameters.]

5. (a) State the definitions of (i) critical region, (ii) Type I error and (iii) Type II error.

Write down the Neyman-Pearson Lemma.

(b) Let

$$f_{\theta_0}(x) = \frac{1}{\sqrt{2\pi}} e^{-x^2/2}, \qquad f_{\theta_1}(x) = \frac{1}{2} e^{-|x|}.$$

Show that given the sample $X_1, \ldots, X_n$, the best test for testing $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$ can be represented in the form,

$$\sum_{i=1}^{n} (|X_i| - 1)^2 > c.$$

In the case $n = 1$, show that the critical region $|X| > 3.1$ gives the best test with significance level $\alpha = 0.002$.

(c) Describe the notion of most accurate (= most selective) confidence interval or set for unknown parameter.

State the theorem on how to construct most accurate confidence intervals (or sets) based on the Most Powerful Test for testing $H_0 : \theta = \theta_0$.

Given a sample $X_1, \ldots, X_n$, construct a one-sided confidence interval of the form $(-\infty, a)$ for θ for $\mathcal{N}(\theta, 1)$ with confidence level $1 - \alpha$. Justify your answer.

END OF QUESTIONS

Normal distribution (areas)

Area ($\alpha = P(Z > z)$) in the tail of the standardized Normal curve, $Z \sim N(0,1)$, for different values of z. Example: Area beyond $z = 1.96$ (or below $z = -1.96$) is $\alpha = 0.02500$. For Normal curve with $\mu = 10$ and $\sigma = 2$, area beyond $x = 12$, say, is the same as the area beyond $z = \dfrac{x-\mu}{\sigma} = \dfrac{12-10}{2} = 1$, i.e. $\alpha = 0.15866$.

z	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.0	.50000	.49601	.49202	.48803	.48405	.48006	.47608	.47210	.46812	.46414
0.1	.46017	.45620	.45224	.44828	.44433	.44038	.43644	.43251	.42858	.42465
0.2	.42074	.41683	.41294	.40905	.40517	.40129	.39743	.39358	.38974	.38591
0.3	.38209	.37828	.37448	.37070	.36693	.36317	.35942	.35569	.35197	.34827
0.4	.34458	.34090	.33724	.33360	.32997	.32636	.32276	.31918	.31561	.31207
0.5	.30854	.30503	.30153	.29806	.29460	.29116	.28774	.28434	.28096	.27760
0.6	.27425	.27093	.26763	.26435	.26109	.25785	.25463	.25143	.24825	.24510
0.7	.24196	.23885	.23576	.23270	.22965	.22663	.22363	.22065	.21770	.21476
0.8	.21186	.20897	.20611	.20327	.20045	.19766	.19489	.19215	.18943	.18673
0.9	.18406	.18141	.17879	.17619	.17361	.17106	.16853	.16602	.16354	.16109
1.0	.15866	.15625	.15386	.15151	.14917	.14686	.14457	.14231	.14007	.13786
1.1	.13567	.13350	.13136	.12924	.12714	.12507	.12302	.12100	.11900	.11702
1.2	.11507	.11314	.11123	.10935	.10749	.10565	.10383	.10204	.10027	.09853
1.3	.09680	.09510	.09342	.09176	.09012	.08851	.08692	.08534	.08379	.08226
1.4	.08076	.07927	.07780	.07636	.07493	.07353	.07215	.07078	.06944	.06811
1.5	.06681	.06552	.06426	.06301	.06178	.06057	.05938	.05821	.05705	.05592
1.6	.05480	.05370	.05262	.05155	.05050	.04947	.04846	.04746	.04648	.04551
1.7	.04457	.04363	.04272	.04182	.04093	.04006	.03920	.03836	.03754	.03673
1.8	.03593	.03515	.03438	.03362	.03288	.03216	.03144	.03074	.03005	.02938
1.9	.02872	.02807	.02743	.02680	.02619	.02559	.02500	.02442	.02385	.02330
2.0	.02275	.02222	.02169	.02118	.02068	.02018	.01970	.01923	.01876	.01831
2.1	.01786	.01743	.01700	.01659	.01618	.01578	.01539	.01500	.01463	.01426
2.2	.01390	.01355	.01321	.01287	.01255	.01222	.01191	.01160	.01130	.01101
2.3	.01072	.01044	.01017	.00990	.00964	.00939	.00914	.00889	.00866	.00842
2.4	.00820	.00798	.00776	.00755	.00734	.00714	.00695	.00676	.00657	.00639
2.5	.00621	.00604	.00587	.00570	.00554	.00539	.00523	.00508	.00494	.00480
2.6	.00466	.00453	.00440	.00427	.00415	.00402	.00391	.00379	.00368	.00357
2.7	.00347	.00336	.00326	.00317	.00307	.00298	.00289	.00280	.00272	.00264
2.8	.00256	.00248	.00240	.00233	.00226	.00219	.00212	.00205	.00199	.00193
2.9	.00187	.00181	.00175	.00169	.00164	.00159	.00154	.00149	.00144	.00139
3.0	.00135	.00131	.00126	.00122	.00118	.00114	.00111	.00107	.00104	.00100
3.1	.00097	.00094	.00090	.00087	.00084	.00082	.00079	.00076	.00074	.00071
3.2	.00069	.00066	.00064	.00062	.00060	.00058	.00056	.00054	.00052	.00050
3.3	.00048	.00047	.00045	.00043	.00042	.00040	.00039	.00038	.00036	.00035
3.4	.00034	.00032	.00031	.00030	.00029	.00028	.00027	.00026	.00025	.00024
3.5	.00023	.00022	.00022	.00021	.00020	.00019	.00019	.00018	.00017	.00017
3.6	.00016	.00015	.00015	.00014	.00014	.00013	.00013	.00012	.00012	.00011
3.7	.00011	.00010	.00010	.00010	.00009	.00009	.00008	.00008	.00008	.00008
3.8	.00007	.00007	.00007	.00006	.00006	.00006	.00006	.00005	.00005	.00005
3.9	.00005	.00005	.00004	.00004	.00004	.00004	.00004	.00004	.00003	.00003
4.0	.00003	.00003	.00003	.00003	.00003	.00003	.00002	.00002	.00002	.00002

α	0.4	0.25	0.2	0.15	0.1	0.05	0.025	0.01	0.005	0.001
z_α	0.2533	0.6745	0.8416	1.0361	1.2816	1.6449	1.9600	2.3263	2.5758	3.0902

END