MATH351201

This question paper consists of 4 printed pages, each of which is identified by the reference **MATH3512**.

Only approved basic scientific calculators may be used

Examination for the Module MATH3512
(May/June 2007)

Viscous Flow

Time allowed: **3 hours**

Do not attempt more than **4** questions. All questions carry equal weight.

1. (a) The equations of motion (Navier-Stokes equations) for a viscous fluid are, in Cartesian co-ordinates (x, y, z),

$$\operatorname{div} \mathbf{u} = 0,$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla\mathbf{u} = \mathbf{X} - \frac{1}{\rho}\nabla p + \nu\nabla^2\mathbf{u},$$

where $\mathbf{u} = (u, v, w)$ is the velocity of the fluid and $\mathbf{X}$ is the body force per unit mass. What are the basic assumptions made in deriving these equations?
What is meant by the *no-slip* boundary condition.

Consider the steady flow U_0 of a viscous fluid, with kinematic viscosity ν, past a body of typical size L. Define the Reynolds number Re for this flow and discuss *briefly* the significance of Re in the flow of a viscous fluid.

(b) A layer of viscous fluid of constant thickness h is in steady, one-dimensional motion down an infinite plane inclined at an angle α to the horizontal. Show that, for this problem, the Navier-Stokes equations reduce to

$$v \equiv 0, \qquad \nu\frac{d^2u}{dy^2} + g\sin\alpha = 0, \qquad \frac{1}{\rho}\frac{dp}{dy} + g\cos\alpha = 0,$$

where x and y measure distance along the plane and normal to it, with u and v being the velocity components in the x and y directions respectively, ν is the coefficient of kinematic viscosity and g the acceleration due to gravity.

Solve the equation for the velocity $u(y)$ in the two cases:
(1) when $y = h$ is a solid surface,
(2) when $y = h$ is a free surface (no shear on the surface).

Calculate the volume flow, defined by $Q = \int_0^h u(y)dy$, in these two cases and show that

$$Q_{(2)} = 4\,Q_{(1)}$$

2. Viscous fluid is in slow, steady two-dimensional motion in the region $0 \le \theta \le \alpha$ between two plates along $\theta = 0$ and $\theta = \alpha$. The plate at $\theta = \alpha$ is fixed and the plate at $\theta = 0$ is moving with a constant velocity U_0 along its length.
The equation governing this slow flow is

$$\nabla^2(\nabla^2\psi) = 0,$$

where ψ is the stream function defined so that the velocity components (u, v) in the r and θ directions are given by

$$u = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad v = -\frac{\partial\psi}{\partial r}$$

and where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

What are the boundary conditions on $\theta = 0$ and $\theta = \alpha$ in terms of the stream function ψ?

Look for a solution in the form

$$\psi(r, \theta) = U_0\, r\, f(\theta),$$

to show that $f(\theta)$ satifies the equation

$$\frac{d^4 f}{d\theta^4} + 2\frac{d^2 f}{d\theta^2} + f = 0.$$

Solve this equation, subject to the boundary conditions, to find $f(\theta)$.

3. Write down the equations for steady thin film flow (lubrication approximation) of a viscous fluid between two surfaces $z = 0$ and $z = h(x)$, stating the assumptions made in deriving these equations.

Suppose that the flow is two-dimensional with the the upper boundary, $z = h(x)$, fixed. The lower surface, $z = 0$, is moving with a constant velocity U_0 along its length. Use your thin film equations to show that the pressure $p(x)$ within the gap between the two surfaces is given by the equation

$$\frac{d}{dx}\left(h^3 \frac{dp}{dx}\right) - 6\mu U_0 \frac{dh}{dx} = 0,$$

where μ is the (constant) coefficient of viscosity of the fluid.

Show that this equation can be integrated to give an expression for $\frac{dp}{dx}$ in terms of $h(x)$.

Suppose that the upper surface has a finite length L, i.e. $h(x)$ is given for $0 \leq x \leq L$, and outside this region the pressure has a constant value p_0. Explain how the equation can be further integrated to find $p(x)$.

If $h(x) = \frac{h_0}{L}(L + x)$ for $0 \leq x \leq L$ (where h_0 is a constant), show that

$$\frac{dp}{dx} = 2\mu U_0 \left(\frac{3}{h(x)^2} - \frac{4h_0}{h(x)^3}\right)$$

and hence that

$$p(x) - p_0 = -\frac{2\mu U_0 L}{h_0^2} \frac{x(L-x)}{(L+x)^2}.$$

4. Explain why it is necessary to introduce a boundary layer to determine the flow past a solid body at large Reynolds number. Use a scaling argument to estimate the thickness of this boundary-layer region in terms of the Reynolds number.

The boundary-layer equations for two-dimensional, steady flow near a solid boundary $(y = 0)$ are

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}, \end{aligned}$$

where $U(x)$ is the outer flow and (u, v) are the velocity components in the x and y directions respectively.

QUESTION 4 CONTINUED...

For the flow near a forward stagnation point, where $U(x) = \dfrac{U_0 x}{l}$, with U_0 and l being velocity and length scales respectively, show that the boundary-layer equations have a similarity solution in the form

$$\psi = \left(\frac{\nu U_0}{l}\right)^{1/2} x f(\eta), \qquad \text{where} \quad \eta = y\left(\frac{U_0}{\nu l}\right)^{1/2}$$

and where ψ is the stream function, defined in the usual way. Determine the equation and the boundary conditions satisfied by $f(\eta)$.

5. Viscous fluid is discharged from a long straight slit in a plane wall so as to form a steady, two-dimensional jet. Cartesian axes are chosen so that the x-axis is the line of symmetry of the jet, with the y-axis normal to it. u and v are the velocity components in the x and y directions respectively. The flow within the jet is assumed to be described by the boundary-layer equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2},$$

subject to the boundary conditions

$$v = 0,\ \frac{\partial u}{\partial y} = 0 \ \text{ on } \ y = 0, \qquad u \to 0 \ \text{ as } \ y \to \pm\infty.$$

By integrating the momentum equation with respect to y, using the continuity equation and applying the boundary conditions, show that

$$\frac{d}{dx}\left(\int_0^\infty u(x,y)^2 dy\right) = 0, \quad \text{and hence} \quad \int_0^\infty u^2(x,y)dy = M, \quad \text{where } M \text{ is a constant.}$$

A similarity solution can be found by expressing the stream function ψ as

$$\psi = (3\nu M)^{1/3} x^{1/3} f(\eta), \quad \text{where} \ \ \eta = \left(\frac{M}{9\nu^2}\right)^{1/3} \frac{y}{x^{2/3}}.$$

Show that, on integrating the equation for $f(\eta)$ with respect to η and applying the boundary conditions,

$$f'' + ff' = 0.$$

Show that $f(\eta) = 2a\tanh(a\eta)$ satisfies this equation and all the required boundary conditions for any (positive) constant a.
Indicate how the integral condition can be used to determine the value of a.

END