MATH351201

This question paper consists of 4 printed pages, each of which is identified by the reference **MATH3512**.

Only approved basic scientific calculators may be used

Examination for the Module MATH3512
(May/June 2006)

Viscous Flow

Time allowed: **3 hours**

Do not attempt more than **4** questions. All questions carry equal weight.

1. (a) The equations of motion (Navier-Stokes equations) for a viscous fluid are, in Cartesian co-ordinates (x, y, z),

$$\operatorname{div} \mathbf{u} = 0,$$

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}.\nabla \mathbf{u} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \mathbf{u},$$

where $\mathbf{u} = (u, v, w)$ is the velocity of the fluid.
What are the assumptions made in deriving these equations?
Describe what is meant by the *no-slip* boundary condition.

(b) Viscous fluid is in steady, two-dimensional motion in a channel between two infinite plates a constant distance h apart. The flow results from a constant applied pressure gradient P and from the plate on $y = 0$ moving with a constant velocity U_0 in the x-direction. Show that the Navier-Stokes equations reduce to

$$\frac{\partial^2 u}{\partial y^2} + \frac{P}{\mu} = 0, \qquad v \equiv 0,$$

where μ is the viscosity of the fluid.

Solve this equation, subject to the appropriate boundary conditions.
Use your solution to show that the volume flow Q in the channel, where $Q = \int_0^h u(y)dy$ is given by

$$Q = \frac{Ph^3}{12\mu} + \frac{U_0 h}{2}.$$

2. Viscous fluid occupying the region $y > 0$ is initially at rest. At time $t = 0$, the fluid is set into motion by a constant shear stress μT_0 being applied in the x-direction along $y = 0$. Show that the Navier-Stokes equations reduce to, for the two-dimensional motion in $y > 0$,

$$\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}, \qquad v \equiv 0$$

where u and v are, respectively, the velocity components in the x and y directions. Show that the initial and boundary conditions for u are

$$\begin{aligned} \frac{\partial u}{\partial y} &= -T_0 \quad \text{on } y = 0, \quad (t > 0) \\ u &\to 0 \quad \text{as } y \to \infty \quad (t > 0) \\ u &= 0 \quad \text{at } t = 0, \quad (y > 0). \end{aligned}$$

Show that the transformation

$$u(y,t) = 2\sqrt{\nu}\, T_0\, t^{1/2} f(\eta), \qquad \eta = \frac{y}{2(\nu t)^{1/2}}$$

reduces the problem to similarity form. Determine the equation and boundary conditions satisfied by $f(\eta)$ and show that this has the solution

$$f(\eta) = \frac{1}{\sqrt{\pi}} \left(\mathrm{e}^{-\eta^2} - 2\eta \int_\eta^\infty \mathrm{e}^{-s^2} ds \right).$$

Hence find the velocity on the surface $y = 0$.

You can assume that $\displaystyle\int_0^\infty \mathrm{e}^{-s^2} ds = \frac{\sqrt{\pi}}{2}$.

3. Viscous fluid occupies the region $0 < \theta < \alpha$. The fluid is in slow, steady two-dimensional motion caused by a plate along $\theta = 0$ moving with a constant velocity U_0 in a direction along its length. $\theta = \alpha$ is a free surface on which the shear stress is zero.
The equation governing this slow flow is

$$\nabla^2(\nabla^2 \psi) = 0,$$

where ψ is the stream function defined so that the velocity components (u, v) in the r and θ directions are given by

$$u = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad v = -\frac{\partial \psi}{\partial r}$$

and where

 QUESTION 3 CONTINUED...

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

By looking for a solution in the form

$$\psi(r,\theta) = U_0 r f(\theta),$$

determine ψ.

Use this result to show that the velocity u_s on the free surface is given by

$$u_s = U_0 \left(\frac{\alpha \cos\alpha - \sin\alpha}{\alpha - \sin\alpha\cos\alpha} \right).$$

Note: You may assume that having zero stress on the surface $\theta = \alpha$ is equivalent to having $\dfrac{\partial^2 \psi}{\partial \theta^2} = 0$ on that boundary.

4. (a) Write down the equations for steady thin film flow of a viscous fluid between the two fixed surfaces $z = 0$ and $z = a$, a constant (small) distance apart (Hele-Shaw cell). State the assumptions made in deriving these equations.
What are the boundary conditions?

Use these equations to find u and v, the velocity components in the x and y directions, satisfying the appropriate boundary conditions.
Hence, show that the vorticity

$$\zeta = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x} = 0.$$

Use your results for u and v to find w, the velocity component in the z direction.
By applying the boundary conditions on w, show that the pressure p satisfies the equation

$$\frac{\partial^2 p}{\partial x^2} + \frac{\partial^2 p}{\partial y^2} = 0.$$

(b) The boundary-value problem

$$\epsilon \frac{d^2 f}{dx^2} + (x+1)\frac{df}{dx} + f = 1, \qquad \text{where } 0 < \epsilon \ll 1$$

on $0 < x < 1$, subject to the boundary conditions

$$f(0) = 0, \quad f(1) = 2,$$

QUESTION 4 CONTINUED...

is to be solved using the method of matched asymptotic expansions.
Show that the solution $f_0(x)$ in the outer region is, at leading order,

$$f_0(x) = \frac{x+3}{x+1}.$$

Show that the scaling for the inner region independent variable ξ is $\xi = x\,\epsilon^{-1}$.
Obtain the leading order solution in the inner region, satisfying the boundary condition on $x = 0$ and matching with the solution in the outer region.

5. (a) Consider the steady flow U_0 of a viscous fluid, with kinematic viscosity ν, past a body of typical size L. Define the Reynolds number Re for this flow.

Explain why it is necessary to introduce a boundary layer to determine the flow past a solid body at large Reynolds number. Use a scaling argument to estimate the thickness δ of this boundary-layer region in terms of the Reynolds number and the length scale L of the body.

Describe *briefly* what is meant by boundary-layer separation.

(b) The boundary-layer equations for two-dimensional, steady flow near a solid boundary ($y = 0$) are

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}, \end{aligned}$$

where $U(x)$ is the outer flow, (u, v) are the velocity components in the x and y directions respectively and ν is the kinematic viscosity.
State the boundary conditions for u and v.

For the flow past a flat plate, where $U(x) = U_0$ is a constant, show that the boundary-layer equations have a similarity solution in the form

$$\psi = (2\nu U_0 x)^{1/2} f(\eta), \qquad \eta = y\left(\frac{U_0}{2\nu x}\right)^{1/2},$$

where ψ is the streamfunction.
Determine the equation and the boundary conditions satisfied by $f(\eta)$.

END