MATH351201

This question paper consists of 4 printed pages, each of which is identified by the reference **MATH3512**.

Only approved basic scientific calculators may be used

Examination for the Module MATH3512
(May/June 2005)

Viscous Flow

Time allowed: **3 hours**

Do not attempt more than **4** questions. All questions carry equal weight.

1. Viscous fluid is in steady, axi-symmetric motion in the annular region between two infinitely long, concentric cylinders $r = a$ and $r = b$, where $a < b$. The flow results from an applied constant pressure gradient $-P$ in the z-direction. Show that the Navier-Stokes equations reduce to

$$\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr} = -\frac{P}{\mu}, \qquad u \equiv 0,$$

where z measures distance along the cylinders and r is the radial co-ordinate. u and w are the velocity components in the r and z directions respectively and μ is the coefficient of viscosity.

State the boundary conditions on $w(r)$ and hence find $w(r)$.

Show that the volume flux Q, given by $Q = 2\pi \int_a^b w(r)\, r\, dr$, is

$$Q = \frac{P\pi}{8\mu}\left(b^2 - a^2\right)\left[b^2 + a^2 - \frac{(b^2 - a^2)}{\log\left(b/a\right)}\right].$$

2. Viscous fluid is in slow, steady two-dimensional motion in the region $-\alpha \leq \theta \leq \alpha$ between two fixed plates along $\theta = -\alpha$ and $\theta = \alpha$. The equation governing this slow flow is

$$\nabla^2(\nabla^2\psi) = 0,$$

where ψ is the stream function defined so that the velocity components (u, v) in the r and θ directions are given by

$$u = \frac{1}{r}\frac{\partial \psi}{\partial \theta}, \qquad v = -\frac{\partial \psi}{\partial r}$$

and where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}.$$

Show that the stream function $\psi(r,\theta)$ for this slow flow in the corner can be found in the form $\psi = r^\lambda f(\theta)$ provided that the constant λ satisfies the equation

$$\lambda \tan(\lambda\alpha) = (\lambda - 2)\tan[(\lambda - 2)\alpha]$$

It can be assumed that ψ is an even function of θ.

In the special case $\alpha = \pi$, find the smallest positive value for λ that satisfies the equation.

3. Write down the equations for steady, two-dimensional thin film flow (lubrication approximation) of a viscous fluid between two surfaces $z = 0$ and $z = h(x)$, stating the assumptions made in deriving these equations.

The upper boundary, $z = h(x)$, is fixed. The lower surface, $z = 0$, is moving with a constant velocity U_0 along its length. Use the thin film equations to show that the pressure $p(x)$ within the gap between the two surfaces is given by the equation

$$\frac{d}{dx}\left(h^3\frac{dp}{dx}\right) - 6\mu U_0\frac{dh}{dx} = 0,$$

where μ is the (constant) coefficient of viscosity of the fluid.

Integrate this equation to obtain an equation for $\frac{dp}{dx}$ in terms of $h(x)$.

The upper surface has a finite length $2L$ with the pressure having a constant value p_0 outside this region. For an upper surface given by $h(x) = \frac{h_0}{L}(2L + x)$ for $-L \le x \le L$, show that the pressure $p(x)$ is given by

$$p(x) - p_0 = -\frac{3\mu U_0 L(L^2 - x^2)}{2h_0^2(2L + x)^2}.$$

4. Consider the steady flow U_0 of a viscous fluid, with kinematic viscosity ν, past a body of typical size L. Define the Reynolds number Re for this flow and discuss *briefly* the significance of Re in the flow of a viscous fluid.

Explain why it is necessary to introduce a boundary layer to determine the flow past a solid body at large Reynolds number. Use a scaling argument to estimate the thickness of this boundary-layer region in terms of the Reynolds number.

Show that, for steady, two-dimensional boundary-layer flow, the vorticity $\zeta = \dfrac{\partial u}{\partial y}$ satisfies the equation

$$\mathbf{u}\cdot\nabla\zeta = \nu\frac{\partial^2\zeta}{\partial y^2}, \qquad \text{where} \;\; \mathbf{u} = (u, v).$$

Explain the significance of the terms in this equation.

Describe what is meant by boundary-layer separation.
Use the boundary-layer equations to show that the skin friction τ_w has a singularity at separation ($x = x_s$) of the form

$$\tau_w \propto (x_s - x)^{1/2} \qquad \text{as} \;\; x \to x_s^- .$$

What is the significance of this singularity for solutions of the boundary-layer equations?

5. The boundary-layer equations for steady, two-dimensional flow near a solid boundary ($y = 0$) are

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}, \end{aligned}$$

where $U(x)$ is the outer flow and (u, v) are the velocity components in the x and y directions respectively.

Show that, for an outer flow $U(x) = a\,x^m$ (a and m constants), the boundary-layer equations have a similarity solution in the form

$$\psi = (a\nu)^{1/2} x^{(m+1)/2} f(\eta), \qquad \text{with} \quad \eta = \left(\frac{a}{\nu}\right)^{1/2} x^{(m-1)/2}\, y,$$

where ψ is the stream function, defined in the usual way, and where $f(\eta)$ satisfies the equation

$$f''' + (\frac{m+1}{2})\, f f'' + m(1 - f'^2) = 0,$$

 QUESTION 5 CONTINUED...

where primes represent differentiation with respect to η.

Show that the further change of variables

$$f = \left(\frac{2}{m+1}\right)^{1/2} F, \qquad \overline{\eta} = \left(\frac{m+1}{2}\right)^{1/2} \eta$$

transforms the equation into

$$F''' + FF'' + \beta(1 - F'^2) = 0 \qquad \text{with} \;\; \beta = \frac{2m}{m+1},$$

where primes now represent differentiation with respect to $\overline{\eta}$.
What are the boundary conditions satisfied by $F(\overline{\eta})$?

To determine how the solution to this equation behaves for β large, we make the transformation

$$F = \beta^{-1/2}\,\phi, \qquad \zeta = \beta^{1/2}\,\overline{\eta}.$$

Show that the equation at leading order for β large is then

$$\phi''' + 1 - \phi'^2 = 0$$

(where primes now represent differentiation with respect to ζ) and that

$$\phi' = 3\tanh^2\left(\frac{\zeta}{\sqrt{2}} + \gamma\right) - 2 \qquad (\text{where } \tanh^2\gamma = 2/3)$$

satisfies the equation and the required boundary conditions.

END