MATH351201

This question paper consists of 4 printed pages, each of which is identified by the reference **MATH3512**.

Only approved basic scientific calculators may be used

Examination for the Module MATH3512
(May/June 2004)

Viscous Flow

Time allowed: **3 hours**

Do not attempt more than **4** questions. All questions carry equal weight.

1. Viscous fluid is in steady, one-dimensional motion in a channel between two infinite fixed plates at a constant distance h apart. The flow results from an applied constant pressure gradient $-P$ in the x-direction and by the plate $y = 0$ moving with speed U_0 in the positive x-direction. Show that the Navier-Stokes equations reduce to

$$\frac{d^2u}{dy^2} = -\frac{P}{\mu}, \qquad v \equiv 0 \qquad\qquad (A)$$

where x and y measure distance along the channel and normal to it, u, and v are the velocity components in the x and y directions respectively and μ is the coefficient of viscosity.

Solve equation (A), subject to the appropriate boundary conditions, to find $u(y)$.

Show that the volume flow Q in the channel is

$$Q = \frac{Ph^3}{12\mu} + \frac{U_0 h}{2} \qquad \text{where} \quad Q = \int_0^h u(y)dy.$$

Show that the shearing stress on the plate $y = 0$ is zero when $U_0 = \dfrac{Ph^2}{2\mu}$.

Sketch the velocity component $u(y)$ in the cases (a) $U_0 < \dfrac{Ph^2}{2\mu}$, (b) $U_0 = \dfrac{Ph^2}{2\mu}$ and (c) $U_0 > \dfrac{Ph^2}{2\mu}$, paying particular attention to where the velocity attains its maximum value.

2. Viscous fluid is in slow, steady two-dimensional motion in the region $0 \le \theta \le \alpha$. There is a fixed plate along $\theta = 0$ and, at $\theta = \alpha$, a constant shearing stress τ_0 is applied. The equation governing this slow flow is

$$\nabla^2(\nabla^2\psi) = 0,$$

where ψ is the stream function defined so that the velocity components (u, v) in the r and θ directions are given by

$$u = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad v = -\frac{\partial\psi}{\partial r}$$

and where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Given that the shear stress $p_{r,\theta}$ in polar co-ordinates is

$$p_{r,\theta} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial\theta}\right),$$

show that this leads to the boundary conditions

$$\psi = 0, \qquad \frac{\partial^2\psi}{\partial\theta^2} = \frac{\tau_0}{\mu}r^2 \qquad \text{on } \theta = \alpha.$$

What are the boundary conditions on $\theta = 0$?

Look for a solution in the form

$$\psi(r,\theta) = \frac{\tau_0}{\mu}r^2 f(\theta),$$

to show that $f(\theta)$ satifies the equation

$$\frac{d^2}{d\theta^2}\left(\frac{d^2 f}{d\theta^2} + 4f\right) = 0.$$

Solve this equation, subject to the boundary conditions, to find $f(\theta)$. Use your result to show that the velocity u_s on the surface $\theta = \alpha$ is

$$u_s = \frac{\tau_0 r}{\mu}\left(\frac{\alpha\sin 2\alpha + \cos 2\alpha - 1}{2\alpha\cos 2\alpha - \sin 2\alpha}\right).$$

3. Write down the equations for steady, two-dimensional thin film flow (lubrication approximation) of a viscous fluid between two surfaces $z = 0$ and $z = h(x)$, stating the assumptions made in deriving these equations.

The upper boundary, $z = h(x)$, fixed. The lower surface, $z = 0$, is moving with a constant velocity U_0 along its length. Use your thin film equations to show that the pressure $p(x)$ within the gap between the two surfaces is given by the equation

$$\frac{d}{dx}\left(h^3\frac{dp}{dx}\right) - 6\mu U_0\frac{dh}{dx} = 0, \qquad (B)$$

where μ is the (constant) coefficient of viscosity of the fluid.

Integrate equation (B) to obtain an equation for $\frac{dp}{dx}$ in terms of $h(x)$.

Suppose that the upper surface has a finite length L, i.e. $h(x)$ is given for $0 \leq x \leq L$, and outside this region the pressure has a constant value p_0. Explain how the equation can be further integrated to find $p(x)$ and the velocity components (or stream function) then determined.

If $h(x) = \frac{h_0}{L}(L + x)$ for $0 \leq x \leq L$ (where h_0 is a constant), show that

$$p(x) - p_0 = \frac{2\mu U_0 L}{h_0^2}\frac{x(x-L)}{(L+x)^2}.$$

4. Consider the steady flow U_0 of a viscous fluid, with kinematic viscosity ν, past a body of typical size L. Define the Reynolds number Re for this flow and discuss *briefly* the significance of Re in the flow of a viscous fluid.

Explain why it is necessary to introduce a boundary layer to determine the flow past a solid body at large Reynolds number. Use a scaling argument to estimate the thickness of this boundary-layer region in terms of the Reynolds number.

The boundary-layer equations for two-dimensional, steady flow near a solid boundary $(y = 0)$ are

$$\begin{aligned}\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0,\\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2},\end{aligned}$$

where $U(x)$ is the outer flow and (u, v) are the velocity components in the x and y directions respectively.

 QUESTION 4 CONTINUED...

For the flow near a forward stagnation point, where $U(x) = \dfrac{U_0 x}{l}$, with U_0 and l being constants, show that the boundary-layer equations have a solution in the form

$$\psi = \left(\frac{\nu U_0}{l}\right)^{1/2} x f(\eta), \qquad \text{where} \qquad \eta = y\left(\frac{U_0}{\nu l}\right)^{1/2}$$

and where ψ is the stream function, defined in the usual way. Determine the equation and the boundary conditions satisfied by $f(\eta)$.

5. Viscous fluid is discharged from a long straight slit in a plane wall so as to form a steady, two-dimensional jet. Cartesian axes are chosen so that the x-axis is the line of symmetry of the jet, with the y-axis normal to it. u and v are the velocity components in the x and y directions respectively. The flow within the jet is assumed to be described by the boundary-layer equations

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2},$$

subject to the boundary conditions

$$v = 0, \ \frac{\partial u}{\partial y} = 0 \ \text{ on } \ y = 0, \qquad u \to 0 \ \text{ as } \ |y| \to \infty.$$

Show that

$$\int_{-\infty}^{\infty} u^2(y)dy = M, \qquad \text{where } M \text{ is a constant.}$$

If the stream function ψ is given by

$$\psi = (3\nu M)^{1/3} x^{1/3} f(\eta), \quad \text{where } \ \eta = \left(\frac{M}{9\nu^2}\right)^{1/3} \frac{y}{x^{2/3}}$$

show that

$$f'' + ff' = 0.$$

Show that $f(\eta) = 2a\tanh(a\eta)$ satisfies this equation and all the required boundary conditions for any (positive) constant a. Use the integral condition to determine the value of a.

END