MATH351201

This question paper consists of 4 printed pages, each of which is identified by the reference **MATH3512**.

Only approved basic scientific calculators may be used

Examination for the Module MATH3512
(May/June 2003)

Viscous Flow

Time allowed: **3 hours**

Do not attempt more than **4** questions. All questions carry equal weight.

1. Viscous fluid is in steady motion in a channel between two infinite fixed plates at a constant distance h apart. The flow results from an applied constant pressure gradient $-P$ and by fluid being withdrawn transversally through the plate $y = 0$ with a constant speed V_w and being injected through the plate $y = h$ at the same rate. Show that the Navier-Stokes equations reduce to

$$\frac{d^2u}{dy^2} + \frac{V_w}{\nu}\frac{du}{dy} = -\frac{P}{\mu}, \qquad v \equiv -V_w$$

for this problem. (x and y measure distance along the channel and normal to it and u, v are the velocity components in the x and y directions respectively.)

Solve this equation, subject to the appropriate boundary conditions, to find $u(y)$.

Show that, by making the transformation

$$u = \frac{Ph}{\rho V_w}U, \qquad Y = \frac{y}{h},$$

the equation becomes in dimensionless form

$$\frac{1}{R}\frac{d^2U}{dY^2} + \frac{dU}{dY} = -1$$

where $R = \dfrac{V_w h}{\nu}$ is the Reynolds number. What are the boundary conditions? Show that this is a singular problem in the limit as $R \to \infty$, with a leading-order solution $U = 1 - Y$ in the outer region. Find the scaling for Y in the boundary layer (inner solution) near $Y = 0$. Hence determine the leading-order term for U in this region.

2. Viscous fluid occupies the region $0 < \theta < \alpha$. The fluid is in slow, steady two-dimensional motion caused by a plate along $\theta = 0$ moving with a constant velocity U_0 in a direction along its length and $\theta = \alpha$ is a free surface on which the shear stress is zero.

The equation governing this slow flow is

$$\nabla^2(\nabla^2\psi) = 0,$$

where ψ is the stream function defined so that the velocity components (u, v) in the r and θ directions are given by

$$u = \frac{1}{r}\frac{\partial\psi}{\partial\theta}, \qquad v = -\frac{\partial\psi}{\partial r}$$

and where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}.$$

Given that the shear stress $p_{r,\theta}$ in polar co-ordinates is given by

$$p_{r,\theta} = \mu\left(r\frac{\partial}{\partial r}\left(\frac{v}{r}\right) + \frac{1}{r}\frac{\partial u}{\partial\theta}\right),$$

show that this leads to the boundary conditions

$$\psi = 0, \qquad \frac{\partial^2\psi}{\partial\theta^2} = 0 \qquad \text{on } \theta = \alpha.$$

What are the boundary conditions on $\theta = 0$?

By looking for a solution in the form

$$\psi(r, \theta) = U_0\, r f(\theta),$$

determine ψ.

Use your result to show that the velocity u_s on the free surface is

$$u_s = U_0\left(\frac{\alpha\cos\alpha - \sin\alpha}{\alpha - \sin\alpha\cos\alpha}\right).$$

3. Write down the equations for steady, two-dimensional thin film flow (lubrication approximation) of a viscous fluid on a fixed surface $z = 0$, stating the assumptions made in deriving these equations.

The upper boundary, $z = h(x)$, is moving with a constant velocity U_0 along its length. Use your thin film equations to show that the pressure p within the gap betwen the two surfaces is given by the equation

$$\frac{d}{dx}\left(h^3 \frac{dp}{dx}\right) + 6\mu U_0 \frac{dh}{dx} = 0,$$

where μ is the (constant) coefficient of viscosity of the fluid.

Write down the equation satisfied by $\dfrac{dp}{dx}$. Explain how this equation can be integrated to find p. If $h(x) = \dfrac{h_1}{L}\sqrt{L^2 + x^2}$ for $-L \leq x \leq L$ (h_1, L constants), express the solution for p in terms of integrals. Any integration constants that appear in the solution should be evaluated explicitly.

4. Consider the steady flow U_0 of a viscous fluid, with kinematic viscosity ν, past a body of typical size L. Define the Reynolds number Re for this flow and discuss briefly the significance of Re in the flow of a viscous fluid.

Explain why it is necessary to introduce a boundary layer to determine the flow past a solid body at large Reynolds number. Use a scaling argument to estimate the thickness of this boundary-layer region in terms of the Reynolds number.

The boundary-layer equations for two-dimensional, steady flow near a solid boundary ($y = 0$) are

$$\begin{aligned} \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} &= 0, \\ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} &= U\frac{dU}{dx} + \nu\frac{\partial^2 u}{\partial y^2}, \end{aligned}$$

where $U(x)$ is the outer flow and (u, v) are the velocity components in the x and y directions respectively.

For the flow past a flat plate, where $U(x) = U_0$, a constant, show that the equations have a similarity solution in the form

$$\psi = (2\nu U_0 x)^{1/2} f(\eta), \quad \text{where} \quad \eta = y\left(\frac{U_0}{2\nu x}\right)^{1/2},$$

 QUESTION 4 CONTINUED...

and where ψ is the stream function, defined in the usual way. Determine the equation and the boundary conditions satisfied by $f(\eta)$.

5. Starting with the boundary-layer equations for steady, two-dimensional flow with an outer flow $U(x)$, derive the momentum integral equation

$$\frac{\tau_w}{\rho U^2} = \frac{d\delta_2}{dx} + \left(\frac{2\delta_2 + \delta_1}{U}\right)\frac{dU}{dx}$$

identifying the various terms in the equation.

Explain why the velocity profile

$$u = \begin{cases} U(x)(2\eta - \eta^2) & (0 \leq \eta \leq 1), \\ U(x) & (\eta > 1) \end{cases}$$

where $\eta = y/\delta$, is a reasonable one to use for an approximate solution of the boundary-layer equations using the momentum integral equation. Derive an equation for $\delta(x)$ from the momentum integral equation and use this to show that

$$\delta(x)^2 = \frac{30\nu}{U(x)^9}\int_0^x U(s)^8 ds.$$

Thus obtain an approximate value for the skin friction for a constant free stream U_0.

END