MATH 350101

This question paper consists of 6 printed pages, each of which is identified by the reference **MATH 3501**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH 3501
(January 2007)

Modelling with Fluids

Time allowed: **2 hours**

You should attempt FOUR questions.
All questions carry equal weight.
Some formulae and identities are given on the final page, these may be useful for some questions.

1. A two-dimensional time-dependent velocity field is given by $\mathbf{u} = (x, x + t)$.

(a) (i) Explain how the streamlines of the flow may be visualised experimentally.

(ii) Find the equation of the streamline passing through the point $(1, 1)$ at time $t = 0$. Sketch the streamline.

(b) (i) Explain how the particle paths may visualised experimentally.

(ii) Find the equation for the path of a particle released into the flow at position $(1, 1)$ at time $t = 0$. Sketch the particle path.

(c) (i) Explain how the streaklines may be visualised experimentally.

(ii) Find the streakline at time $t = 0$ for dye released into the flow at position $(1, 1)$ over the previous 10 time-units. Sketch the streakline.

(d) Under what circumstances do the streamlines, pathlines and streaklines coincide?

(e) Write down an expression for the Lagrangian acceleration in a flow. Calculate the acceleration felt by a particle in this flow.

2. A two-dimensional vortex flow is given in Cartesian co-ordinates by

$$u = -\frac{\kappa y}{(x-a)^2 + y^2} \quad \text{and} \quad v = \frac{\kappa (x-a)}{(x-a)^2 + y^2}.$$

Show that the velocity field is incompressible. Find the streamfunction and sketch the streamlines.

Calculate the vorticity of the flow. Show that the velocity potential for this flow is

$$\phi = \kappa \tan^{-1}\left(\frac{y}{x-a}\right).$$

[Note: $\frac{\mathrm{d}}{\mathrm{d}z} \tan^{-1} z = \frac{1}{1+z^2}$.]

A wall is located at $x = 0$. State the boundary condition that the flow must satisfy on the wall. By adding an image vortex, or otherwise, find the velocity potential in this case. Find the velocity field and check that the boundary condition is satisfied. Sketch the flow.

3. Starting from the Euler momentum equation for an inviscid incompressible fluid in the absence of body forces

$$\rho\left(\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u}\right) = -\nabla p,$$

where $\mathbf{u}$ is the fluid velocity, ρ is the fluid density and p is the pressure, show that for steady flows

$$\mathcal{H} = \frac{p}{\rho} + \frac{|\mathbf{u}|^2}{2}$$

is constant on streamlines.

Such a fluid flows steadily through a long cylindrical elastic tube having circular cross-section. The variable z measures distance downstream along the axis of the tube. The tube wall has thickness $h(z)$, so that if the external radius of the tube is $r(z)$, its internal radius is $r(z) - h(z)$, where $h(z) \geq 0$ is a given slowly-varying function that tends to zero as $z \to \pm\infty$. The elastic tube wall exerts a pressure $p(z)$ on the fluid given as

$$p(z) = p_0 + k\left(r(z) - R\right)$$

where p_0, k and R are all positive constants. Far upstream, r has the constant value R, the fluid pressure has the constant value p_0 , and the fluid velocity u has the constant value V. Assume that gravity is negligible and that $h(z)$ varies sufficiently slowly that the velocity may be taken as uniform across the tube at each value of z.

Use conservation of mass to show that

$$r(z) = R\left(\frac{V}{u(z)}\right)^{1/2} + h(z).$$

Now apply Bernoulli's equation to show that

$$\frac{h}{R} = 1 - \left(\frac{V}{u}\right)^{1/2} + \frac{\lambda}{4}\left(1 - \frac{u^2}{V^2}\right),$$

where $\lambda = 2\rho V^2/kR$. Sketch h/R against u/V.

Show that if h is greater than some critical value $h_c(\lambda)$ then no flow is possible. Find $h_c\left(\lambda\right)$.

4. A parallel flow in the z-direction is incident on a sphere centred at the origin and of radius a. Assume that the fluid flow is irrotational.

State the boundary condition that must be satisfied on the surface of the sphere. Take the velocity potential of the form

$$\phi = Uz - \frac{\mu z}{(r^2 + z^2)^{3/2}}$$

in cylindrical polar coordinates (r, θ, z). Calculate the value of μ such that the boundary condition on the surface of the sphere is satisfied.

Find the Stokes streamfunction Ψ and draw the streamlines for $r > a$ and also $r < a$. Briefly explain the role of the flow inside the sphere $r < a$.

5. Water from a large deep reservoir flows over a weir. Distance down the channel of fluid is given by x. The water is of depth $d(x)$ when the surface of the water has fallen a distance $h(x)$ below that far upstream in the reservoir. Assume that the depth of water varies sufficiently slowly that the velocity can be taken to be horizontal and uniform in depth. The breadth of the channel B is constant.

State Bernoulli's equation and apply this to the surface streamline to find the fluid velocity. Hence, show that the volume flux of fluid is

$$Q = Bd\sqrt{2gh}.$$

Explain why Q does not vary down the channel. Assuming that $h + d$ has a minimum at the crest of the weir, show that $h = d/2$ at the crest of the weir.

Show that

$$Q^2 = \frac{8B^2L^3g}{27}$$

where L is the minimum value of $h + d$.

Far downstream the fluid surface has dropped $8L$ below the upstream height and the fluid depth has become constant. Find the depth of fluid far downstream.

Define the Froude number for the flow and give a physical interpretation of the Froude number. Calculate the Froude number at the crest of the weir. Is the flow over the weir a critical flow?

Cylindrical polars (r, θ, z)

For a scalar field $p(r, \theta, z)$ and a vector field $\mathbf{u} = (u_r, u_\theta, u_z)$

$$
\begin{aligned}
\nabla p &= \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{\partial p}{\partial z}\mathbf{e}_z \\
\nabla \cdot \mathbf{u} &= \frac{1}{r}\frac{\partial}{\partial r}(r u_r) + \frac{1}{r}\frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \\
\nabla \times \mathbf{u} &= \left(\frac{1}{r}\frac{\partial u_z}{\partial \theta} - \frac{\partial u_\theta}{\partial z}\right)\mathbf{e}_r + \left(\frac{\partial u_r}{\partial z} - \frac{\partial u_z}{\partial r}\right)\mathbf{e}_\theta + \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right)\mathbf{e}_z \\
\nabla^2 p &= \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial p}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 p}{\partial \theta^2} + \frac{\partial^2 p}{\partial z^2}
\end{aligned}
$$

Spherical polars (r, θ, ϕ)

For a scalar field $p(r, \theta, \phi)$ and a vector field $\mathbf{u} = (u_r, u_\theta, u_\phi)$

$$
\begin{aligned}
\nabla p &= \frac{\partial p}{\partial r}\mathbf{e}_r + \frac{1}{r}\frac{\partial p}{\partial \theta}\mathbf{e}_\theta + \frac{1}{r\sin\theta}\frac{\partial p}{\partial \phi}\mathbf{e}_\phi \\
\nabla \cdot \mathbf{u} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 u_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}(u_\theta \sin\theta) + \frac{1}{r\sin\theta}\frac{\partial u_\phi}{\partial \phi} \\
\nabla \times \mathbf{u} &= \frac{1}{r\sin\theta}\left(\frac{\partial}{\partial \theta}(u_\phi \sin\theta) - \frac{\partial u_\theta}{\partial \phi}\right)\mathbf{e}_r + \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial u_r}{\partial \phi} - \frac{\partial}{\partial r}(r u_\phi)\right)\mathbf{e}_\theta \\
&\quad + \frac{1}{r}\left(\frac{\partial}{\partial r}(r u_\theta) - \frac{\partial u_r}{\partial \theta}\right)\mathbf{e}_\phi \\
\nabla^2 p &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial p}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial p}{\partial \theta}\right) + \frac{1}{r^2\sin\theta}\frac{\partial^2 p}{\partial \phi^2}
\end{aligned}
$$

Vector identities

$$
\begin{aligned}
\nabla \times \nabla p &= 0 \\
\nabla \cdot (\nabla \times \mathbf{u}) &= 0 \\
\nabla \cdot (p\mathbf{u}) &= p\nabla \cdot \mathbf{u} + \mathbf{u} \cdot \nabla p \\
\nabla^2 \mathbf{u} &= \nabla(\nabla \cdot \mathbf{u}) - \nabla \times (\nabla \times \mathbf{u}) \\
(\nabla \times \mathbf{u}) \times \mathbf{u} &= \mathbf{u} \cdot \nabla \mathbf{u} - \frac{1}{2}\nabla|\mathbf{u}|^2
\end{aligned}
$$

END