MATH 343301

This question paper consists of 4 printed pages, each of which is identified by the reference **MATH 3433**.

Only approved basic scientific calculators may be used.

Examination for the Module MATH 3433
(January 2007)

Electromagnetism

Time allowed: **3 hours**

Do not answer more than FOUR questions. All questions carry equal weight.

The following vector identities may be helpful.
For vector fields $\mathbf{a}$ and $\mathbf{b}$ and a scalar field f:

$$\nabla \times (\nabla \times \mathbf{a}) = \nabla(\nabla \cdot \mathbf{a}) - \nabla^2 \mathbf{a}$$

$$\nabla \cdot (f\mathbf{a}) = f(\nabla \cdot \mathbf{a}) + \mathbf{a} \cdot \nabla f$$

$$\nabla \times (f\mathbf{a}) = \nabla f \times \mathbf{a} + f \nabla \times \mathbf{a}$$

$$\nabla \times (\mathbf{a} \times \mathbf{b}) = (\mathbf{b} \cdot \nabla)\mathbf{a} - \mathbf{b}(\nabla \cdot \mathbf{a}) - (\mathbf{a} \cdot \nabla)\mathbf{b} + \mathbf{a}(\nabla \cdot \mathbf{b})$$

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1. State clearly Coulomb's law describing the force between two electric point charges at rest. Explain how the idea of an *electric field* is related to the force exerted by a distribution of electric charges.

State the relation between the electric field $\mathbf{E}$ and the electrostatic potential ϕ.

Write down the electric field due to a point charge q situated at the origin, and hence write down the electrostatic potential.

A point charge q is located at the point $(0, 0, h)$ (in Cartesian coordinates) above an infinite plane conductor located at $z = 0$ and maintained at zero potential. Use the method of images to calculate the potential in the region $z > 0$. Calculate the electric field in $z > 0$ and sketch the field lines. Explain how the surface charge density σ on the surface $z = 0$ is related to the electric field. Hence calculate σ. Check your answer by integrating σ over the surface $z = 0$.

Now suppose that a point charge is placed at an arbitrary position within a wedge-shaped region, the walls of which are perfect conductors maintained at zero potential, subtend an angle of $\pi/3$ and extend to infinity. Draw a sketch to show where the image charges should be placed in order to solve this problem by the method of images.

2. A spatially bounded charge distribution occupying a volume V has charge density ρ and electric scalar potential ϕ. Show that the electrostatic energy W can be written as

$$W = \frac{1}{2}\int_V \rho\phi dV. \tag{1}$$

Show, by using the equations of electrostatics, that this may be written as

$$W = \frac{\epsilon_0}{2}\int \mathbf{E}^2 dV, \tag{2}$$

where the integral is taken over all space.

Total charge Q is uniformly distributed throughout the spherical annulus $a < r < b$. Explain why the electric field takes the form $\mathbf{E} = (E(r), 0, 0)$ in spherical polar coordinates. State Gauss's law. Use it to calculate E in the three regions $r < a$, $a < r < b$ and $r > b$. Hence show that the potential ϕ is given by:

$$\begin{aligned}
\phi &= \frac{Q}{4\pi\epsilon_0 r} \quad \text{for} \quad r > b; \\
\phi &= \frac{Q}{8\pi\epsilon_0(b^3 - a^3)}\left(3b^2 - r^2 - \frac{2a^3}{r}\right) \quad \text{for} \quad a < r < b; \\
\phi &= \frac{3Q}{8\pi\epsilon_0(b^3 - a^3)}\left(b^2 - a^2\right) \quad \text{for} \quad r < a.
\end{aligned}$$

Use either expression (1) or expression (2) to show that the electrostatic energy is given by

$$W = \frac{3Q^2(2b^5 - 5b^2a^3 + 3a^5)}{40\pi\epsilon_0(b^3 - a^3)^2}\,.$$

3. What is the physical significance of the equation

$$\nabla \cdot \mathbf{B} = 0, \tag{1}$$

where $\mathbf{B}$ is the magnetic field? Explain how the result (1) implies the existence of a vector potential $\mathbf{A}$. Explain why the vector potential is not uniquely defined.

Write down the equation governing the magnetic field and the current density $\mathbf{j}$ for the case of steady current flow. Show that, under a certain condition on $\mathbf{A}$, which should be stated, $\mathbf{A}$ and $\mathbf{j}$ are related by the equation

$$\nabla^2\mathbf{A} = -\mu_0\mathbf{j}. \tag{2}$$

Suppose that a steady current flows in a bounded conductor occupying a volume V. By comparison of (2), in Cartesian coordinates, with a similar equation that arises in electrostatics, show that the vector potential $\mathbf{A}$ at a point P with position vector $\mathbf{r}$, lying outside V, is given by

$$\mathbf{A} = \frac{\mu_0}{4\pi}\int_V \frac{\mathbf{j}(\mathbf{r}')}{|\mathbf{r}' - \mathbf{r}|} d^3\mathbf{r}'\,.$$

Hence show that the magnetic field $\mathbf{B}$ at P is given by

$$\mathbf{B} = \frac{\mu_0}{4\pi} \int_V \frac{(\mathbf{r}' - \mathbf{r}) \times \mathbf{j}}{|\mathbf{r}' - \mathbf{r}|^3} d^3\mathbf{r}' \,. \tag{3}$$

If the current flows only in a closed wire loop C, with line element $\mathbf{ds}$, and has constant magnitude I, show that equation (3) becomes

$$\mathbf{B} = \frac{\mu_0 I}{4\pi} \oint_C \frac{(\mathbf{r}' - \mathbf{r}) \times \mathbf{ds}}{|\mathbf{r}' - \mathbf{r}|^3}. \tag{4}$$

A steady current I flows round a circular wire loop of radius a, centred at the origin and located in the plane $z = 0$. Using expression (4), show that the magnetic field at the point with coordinates $(0, 0, z)$ is

$$\mathbf{B} = \frac{\mu_0 I a^2}{2(a^2 + z^2)^{3/2}} \,\hat{\mathbf{z}}.$$

4. State Ohm's law relating the current and the electric field in a conductor. Use Maxwell's equations to deduce that in a conductor with permittivity ϵ, permeability μ and conductivity σ (all constants):

$$\nabla^2 \mathbf{B} = \mu\sigma \frac{\partial \mathbf{B}}{\partial t} + \mu\epsilon \frac{\partial^2 \mathbf{B}}{\partial t^2}.$$

A plane electromagnetic wave with electric and magnetic fields given, in complex notation, by

$$\mathbf{E} = \mathbf{E}_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)}, \quad \mathbf{B} = \mathbf{B}_0 e^{i(\mathbf{k}\cdot\mathbf{x} - \omega t)},$$

travels through such a conductor. Show that ω and k $(= |\mathbf{k}|)$ are related by the expression

$$k^2 = \epsilon\mu\omega^2 \left(1 + \frac{i\sigma}{\epsilon\omega}\right).$$

By obtaining an approximate expression for k for the case of a very poor conductor $(\sigma \ll \epsilon\omega)$, show that, to first order, the attenuation of waves in a poor conductor is independent of frequency. Now obtain the leading order approximation for k for the case of a *good* conductor $(\sigma \gg \epsilon\omega)$. Hence show that the skin depth δ is given by

$$\delta = \sqrt{\frac{2}{\mu\sigma\omega}}.$$

5. Show, from Maxwell's equations, that in free space the electric field $\mathbf{E}$ and the magnetic field $\mathbf{B}$ satisfy the wave equation, namely

$$\frac{\partial^2 \mathbf{E}}{\partial t^2} = c^2 \nabla^2 \mathbf{E}, \qquad \frac{\partial^2 \mathbf{B}}{\partial t^2} = c^2 \nabla^2 \mathbf{B},$$

where c is the speed of light.

A plane electromagnetic wave is normally incident from free space, occupying the half-space $z < 0$, onto the plane face of a semi-infinite block of glass, with permittivity ϵ_1 and permeability μ_1, occupying the half-space $z > 0$. The electric field in $z < 0$ consists of an incident and reflected wave in the form

$$\mathbf{E} = \left(I e^{i(k_0 z - \omega t)} + R e^{i(-k_0 z - \omega t)} \right) (1, 0, 0).$$

The transmitted wave in $z > 0$ takes the form

$$\mathbf{E} = \left(T e^{i(k_1 z - \omega t)} \right) (1, 0, 0).$$

What is the relation between k_0 and ω and between k_1 and ω? Use Faraday's law to calculate the magnetic field in $z < 0$ and $z > 0$. Derive the boundary conditions that $\mathbf{E}$ and $\mathbf{H}$ must satisfy on $z = 0$, assuming there is no surface charge or surface current. Use these to obtain the ratios T/I and R/I in terms of ϵ_0, ϵ_1, μ_0 and μ_1.

END