MATH–339501

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–339501

Only approved basic scientific calculators may be used.

Examination for the Module MATH–3395

(May/June 2007)

Dynamical Systems

Time allowed: 3 hours

Answer **four** questions.
All questions carry equal marks.

1. Consider the map defined by

$$f(x) = \mu x(1-x)^2$$

where $\mu \geq 0$ is a parameter.

(a) Find the maximum of the map in the interval $I = [0, 1]$, and hence find the range of values of μ for which $f(x)$ is a map from I to itself.

(b) Suppose that x^* is a fixed point of the map f. State the condition that x^* satisfies, and the conditions under which x^* is a stable fixed point.

(c) Suppose that $\mu = 2.25$.

(i) Find the fixed points that are in I, and state whether or not they are stable.

(ii) Sketch the map for $\mu = 2.25$, including the diagonal on your sketch.

(iii) Calculate the first four iterates of the orbit starting from initial condition $x_0 = 0.2$, and include this orbit as a cobweb diagram on your sketch. What do you think will happen to orbits starting from other values of x_0?

(d) For general values of μ in the range for which $f(x)$ is a map from I to itself, find the fixed points of the map, and state the range of μ for which the fixed points are in I and are stable.

2. **(a)** Define what it means for a continuous map $f(x)$ from an interval I to itself to have a *horseshoe*. Define what it means for a continuous map to be *topologically chaotic*.

(b) Suppose that a continuous map $f(x)$ from an interval I to itself has a periodic orbit of least period three. Prove that $f^2(x)$ has a horseshoe.

continued ...

3. Suppose that two differentiable maps $x \to f(x)$ and $y \to g(y)$ are *conjugate*, that is, there is a differentiable invertible coordinate transformation $y = h(x)$ such that

$$g(h(x)) = h(f(x))$$

for all x.

(a) Suppose that x^* is a fixed point of f. Show that $y^* = h(x^*)$ is a fixed point of g.

(b) Given two initial conditions x_0 and $y_0 = h(x_0)$, show that $y_n = h(x_n)$, where $x_{j+1} = f(x_j)$ and $y_{j+1} = g(y_j)$.

(c) The *Lyapunov exponent* $L_f(x_0)$ of an orbit, with initial condition x_0, for the map f is defined as:

$$L_f(x_0) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} \log |f'(x_j)|$$

Show that $L_f(x_0) = L_g(y_0)$, with $y_0 = h(x_0)$.

(d) Show that the Logistic map $f(y) = 4y(1-y)$ and the tent map

$$T(x) = \begin{cases} 2x & 0 \le x \le \frac{1}{2}, \\ 2(1-x) & \frac{1}{2} \le x \le 1 \end{cases}$$

are conjugate by taking $h(x) = \sin^2\left(\frac{1}{2}\pi x\right)$.

(e) Hence show that the Lyapunov exponent for a chaotic orbit in the Logistic map is equal to $\log 2$.

4. **(a)** Define what it means for a map $f(x)$ from an interval I to itself to have *Sensitive Dependence* at a point x_0 with sensitivity constant C. Define what it means for the map to have *Sensitive Dependence on Initial Conditions* (SDIC).

(b) Consider the Shift map $S(x) = 2x \bmod 1$, defined from $I = [0, 1)$ to itself.

(i) Suppose that x_0 is a pre-image of 0, that is, $S^n(x_0) = 0$ for some $n > 0$. State what this implies about the binary representation of x_0. Show that the Shift map has Sensitive Dependence at x_0, specifying the value of the sensitivity constant.

(ii) Suppose that x_0 is not a pre-image of 0, that is, $S^n(x_0) \neq 0$ for any integer n. State what this implies about the binary representation of x_0. Show that the Shift map has Sensitive Dependence at x_0, specifying the value of the sensitivity constant.

(iii) Hence show that the Shift map has Sensitive Dependence on Initial Conditions.

Question 4 continues . . .

(c) Suppose that $f(x)$ is a symmetric one-hump map, that is: $f(0) = f(1) = 0$, $f(\frac{1}{2}) = 1$, $f(x) = f(1-x)$, and f is monotonic increasing on $(0, \frac{1}{2})$ and monotonic decreasing on $(\frac{1}{2}, 0)$.

(i) State the conditions that guarantee that f has Sensitive Dependence on Initial Conditions.

(ii) Hence show that the Logistic map $f(x) = 4x(1-x)$ has Sensitive Dependence on Initial Conditions, explaining carefully how the map satisfies each of the required conditions. You may use the fact that the Schwartzian derivative is defined by

$$Sf(x) = \frac{f'''(x)}{f'(x)} - \frac{3}{2}\left(\frac{f''(x)}{f'(x)}\right)^2 .$$

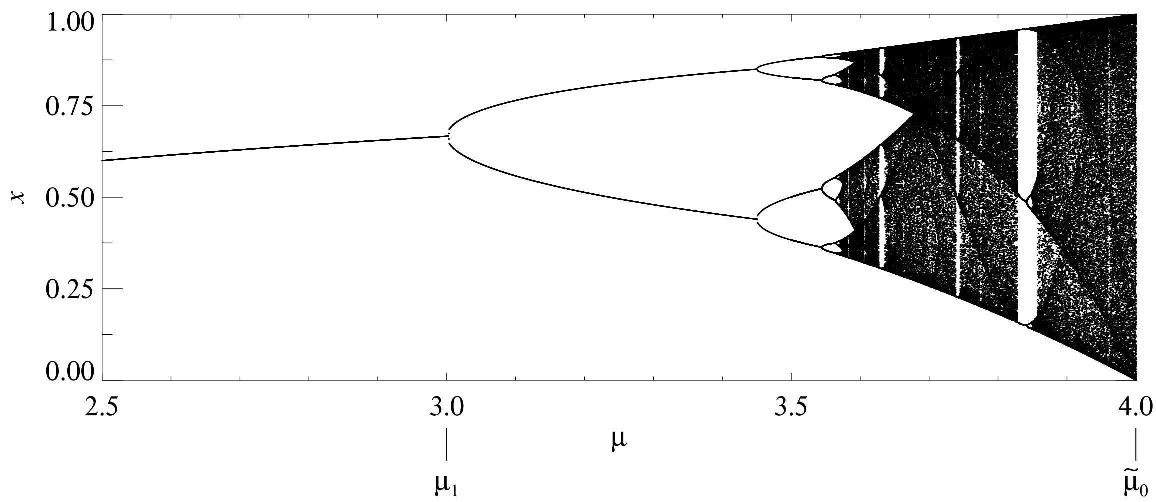
5. Consider the Logistic map $f(x) = \mu x(1-x)$ from $I = [0, 1]$ to I, with $0 \le \mu \le 4$.

(a) Find the fixed points of the map f and calculate their stability. State clearly for which ranges of μ the fixed points are stable, and identify the type of bifurcation at which the fixed points lose their stability.

(b) Sketch $f(x)$ against x for parameter values $\mu_0 = 1$, $\mu_1 = 3$ and $\tilde{\mu}_0 = 4$, including the diagonal in your sketch and indicating clearly whether the slope of f is equal to 1 or -1 at the fixed points.

(c) Sketch $f^2(x)$ against x for $\mu_1 = 3$ and $\tilde{\mu}_0 = 4$, including the diagonal and slopes as above. Explain clearly why there must be two intermediate values of μ: $3 < \mu_2 < \tilde{\mu}_1 < 4$ such that when $\mu = \mu_2$, there is a period-doubling bifurcation from a period-two orbit, and when $\mu \ge \tilde{\mu}_1$, the map f^2 has a horseshoe. Sketch f^2 at μ_2 and $\tilde{\mu}_1$.

(d) Discuss how the ideas above help explain the period-doubling cascade and the inverse period-doubling cascade in the Logistic map (see figure below).



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