MATH–339501

This question paper consists of 3 printed pages, each of which is identified by the reference MATH–339501

Only approved basic scientific calculators may be used.

Examination for the Module MATH–3395

(May/June 2006)

Dynamical Systems

Time allowed: 3 hours

Answer **four** questions.
All questions carry equal marks.

1. Suppose that $f(x)$ is a differentiable map from an interval I to itself.

(a) Explain what is meant by a *fixed point* of the map, by a *point of least period 2*, and by an *orbit of least period 2*.

(b) Suppose that the derivative $|f'(x)| \neq 1$ at all of the fixed points of the map in I, and let $R(x)$ be defined by

$$R(x) = \frac{f^2(x) - x}{f(x) - x}.$$

(i) Show (for example, by using l'Hôpital's rule) how $R(x)$ can be evaluated at the fixed points of f.

(ii) Suppose that x^* is a fixed point of f. Show that $R(x^*) \neq 0$.

(iii) Hence show that the roots of $R(x)$ are points of least period 2, and show that the number of such roots must be even.

(c) Suppose that $f(x) = \mu x(1-x)$, where $0 \leq \mu \leq 4$, and $I = [0,1]$.

(i) Find the fixed points of the map, specifying the range of values of μ over which these points exist and are in I.

(ii) Show that $R(x)$ can be simplified to a quadratic function of x, and hence find $R(x)$.

(iii) Find the points of least period 2, specifying the range of values of μ over which these points exist and are in I.

(iv) Identify the bifurcations from the fixed points of f.

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2. Suppose that $f(x)$ is a map from an interval I to itself, and that $x^* \in I$ is a fixed point of f.

(a) Define what it means for x^* to be *Lyapunov stable*.

(b) Define what it means for x^* to be *asymptotically stable*.

(c) Define what it means for x^* to be *stable*.

(d) Give two example maps (either by an explicit formula or by a sketch) where x^* is

(i) asymptotically stable but not Lyapunov stable;

(ii) Lyapunov stable but not asymptotically stable.

In both cases, explain carefully how your example has the required properties.

(e) Suppose now that f is differentiable at x^*. Prove that if $|f'(x^*)| < 1$, then x^* is stable, and that all orbits starting close enough to x^* converge to x^*. State and prove carefully all intermediate results that you need.

3. **(a)** Define what it means for a continuous map $f(x)$ from an interval I to itself to have a *horseshoe*. Define what it means for a continuous map to be *topologically chaotic*. Give an example of a map that has a horseshoe, explaining carefully why it does.

(b) Define the ω*-limit set* of an initial condition x_0. Define what is meant by a *dense orbit*, and give the ω-limit set of a point on a dense orbit. Give an example of a map that has a dense orbit, specifying the initial condition of the orbit.

(c) Suppose a continuous map $f(x)$ has a horseshoe.

(i) Explain what this implies about the orbits of f, and explain which orbits of f can be put into correspondence with the orbits of the Shift map $S(x) = 2x \bmod 1$.

(ii) Explain why such a map f might or might not have a dense orbit.

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4. Consider the family of tent maps:

$$x_{n+1} = T_\mu(x_n) = \begin{cases} \mu x_n & \text{if } 0 \le x_n \le \frac{1}{2} \\ \mu(1 - x_n) & \text{if } \frac{1}{2} \le x_n \le 1 \end{cases}$$

(a) Show that T_μ is a map from $I = [0, 1]$ to itself provided that $0 \le \mu \le 2$.

(b) Show that if $0 \le \mu < 1$, all orbits tend to zero.

(c) Explain what happens when $\mu = 1$.

(d) Show that the map T_μ has a horseshoe if and only if $\mu = 2$.

(e) Show that if $\sqrt{2} \le \mu \le 2$, the map T_μ^2 has a horseshoe.

(f) Show that there is a linear transformation $y = h(x) = ax + b$ (where a and b are to be found) such that

$$h\left(T_\mu^2(x)\right) = T_{\mu^2}\left(h(x)\right)$$

for all $x \in J$, where $J \subset I$ is a subinterval that should be specified.

(g) Explain how this allows one to conclude that T_μ is chaotic for all parameter values in the interval $1 < \mu \le 2$.

5. **(a)** Define the *topological entropy* $h(f)$ of a continuous map $f(x)$.

(b) Suppose that a continuous map $f(x)$ from an interval I to itself has a periodic orbit of least period three. Prove that the topological entropy of the map is at least $\log\left(\frac{1+\sqrt{5}}{2}\right)$.

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