MATH–339501

This question paper consists of 2 printed pages, each of which is identified by the reference MATH–339501

Only approved basic scientific calculators may be used.

Examination for the Module MATH–3395

(May/June 2005)

Dynamical Systems

Time allowed: 2 hours

Answer **four** questions.
All questions carry equal marks.

1. Find all fixed points and period-two orbits of the following maps. In each case, classify the fixed points and periodic orbits as stable, unstable or undetermined. Sketch a graph of each map (including the diagonal in your sketch). Calculate the first five iterates starting from the initial condition $x_0 = 0.5$, and plot the trajectory of this initial condition as a cobweb diagram.

(a) $f(x) = 3x(1-x)$, defined from $I = [0, 1]$ to itself.

(b) $f(x) = \text{sgn}(x)(-1 + 2x^2)$, defined from $I = [-1, 1]$ to itself, where $\text{sgn}(x) = 1$ if $x > 0$, $\text{sgn}(x) = 0$ if $x = 0$, and $\text{sgn}(x) = -1$ if $x < 0$.

(c) $f(x) = -2x + \frac{1}{3}x^3$, defined from $I = [-3, 3]$ to itself.

Note: in (b) and (c), you need only look for period-two points x^* that satisfy $f(x^*) = -x^*$.

2. **(a)** Define what it means for a continuous map $f(x)$ from an interval I to itself to have a *horseshoe*. Define what it means for a continuous map to be *topologically chaotic*. Give an example of a map that has a horseshoe, explaining carefully why it does.

(b) Suppose two maps $f(x)$ and $g(x)$, both from an interval I to itself, are *conjugate*, that is, there is a continuous invertible map $h(x)$ such that $f(h(x)) = h(g(x))$ for all $x \in I$.

(i) Show that if x^* is a period-n point of g, then $h(x^*)$ is a period-n point of f.

(ii) Show that if g has a horseshoe, then f also has a horseshoe.

3. **(a)** Define the *topological entropy* $h(f)$ of a continuous map $f(x)$.

(b) Prove that $h(f^m) = mh(f)$, where m is a positive integer.

(c) Consider the generalised shift map $S_3(x) : I \to I$, defined by

$$S_3(x) = 3x \bmod 1,$$

where $I = [0, 1)$. Plot $S_3(x)$ and the second iterate $S_3^2(x)$. Compute the orbit of $x_0 = \frac{1}{7}$.

(d) Compute the topological entropy $h(S_3)$.

4. Suppose that a continuous map $f(x)$ from an interval I to itself has a periodic orbit of period three. Prove that $f(x)$ has periodic orbits of all periods, without using Sharkovsky's Theorem.

5. Consider the Logistic map $f(x) = \mu x(1 - x)$ from $I = [0, 1]$ to I, with $0 \leq \mu \leq 4$.

(a) Find the fixed points of the map f and calculate their stability. Plot a bifurcation diagram of x against μ, for $0 \leq \mu \leq 4$, identifying clearly the bifurcation points.

(b) Sketch $f(x)$ against x for parameter values $\mu_0 = 1$, $\mu_1 = 3$ and $\tilde{\mu}_0 = 4$, including the diagonal in your sketch and indicating clearly whether the slope of f is equal to 1 or -1 at the fixed points.

(c) Sketch $f^2(x)$ against x for $\mu_1 = 3$ and $\tilde{\mu}_0 = 4$, including the diagonal and slopes as above. Explain clearly why there must be two intermediate values of μ: $3 < \mu_2 < \tilde{\mu}_1 < 4$ such that when $\mu = \mu_2$, there is a period-doubling bifurcation from a period-two orbit, and when $\mu \geq \tilde{\mu}_1$, the map f^2 has a horseshoe. Sketch f^2 at μ_2 and $\tilde{\mu}_1$.

(d) Discuss how the ideas above help explain many features of the bifurcation diagram of the Logistic map, particularly the period-doubling cascade and the inverse period-doubling cascade.

END